Extra Qu-s

$\lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)} = 27$

Concept 19
Know the limit of k as x tends to a is k

Example 3 If k is a constant, find $\lim_{x\to 9}(k)$

Solution $p(x) = k$ is a polynomial in x of degree zero. Thus, $\lim_{x\to 9}(k) = \lim_{x\to 9} p(x) = p(9) = k$

Concept 20
Application on finding the limit of a rational expression by "canceling zeros"

Example 4 Find $\lim_{x\to 2}\frac{x-2}{x^2-4}$

Solution Since the denominator of the given expression is zero when $x = 2$ then the quotient rule for evaluating limits does not apply here.

For $x \neq 2$, $\frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2}$. Thus, $\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{4}$.

Example 5 Show that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Solution Since the denominator of the given expression is zero when $\theta = 0$ then the quotient rule for evaluating limits does not apply.

In figure 1.2.2(a), the area of ΔOPA is less than the area of sector OPA which, in turns, is less than the area of ΔOTA.

In terms of θ, noting that the radius of the circle is 1, $PH = \sin\theta$, and $AT = \tan\theta$. This translates to

$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta,\ 0 < \theta < \frac{\pi}{2}$.

Dividing all three sides by $\frac{1}{2}\sin\theta$

yields, $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

Taking limits as $\theta \to 0^+$ gives,

$1 \le \lim_{\theta\to 0^+}\frac{\theta}{\sin\theta} \le 1$. By the Sandwich theorem, $\lim_{\theta\to 0^+}\frac{\theta}{\sin\theta} = 1$.

$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0^+}\frac{1}{\frac{\theta}{\sin\theta}} = \frac{1}{1} = 1$.

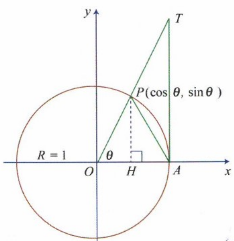

Figure 1.2.2(a) The unit circle with θ in quadrant one.

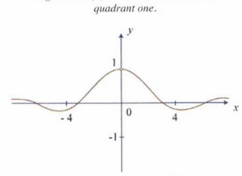

Figure 1.2.2(b) The graph of $y = \frac{\sin x}{x}$

A similar argument can be used to show that for $-\frac{\pi}{2} < \theta < 0$, $\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1$.

Therefore, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. Note that, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ is valid only if θ is in radians.

In the example above, $\lim_{x\to a^+}$ and $\lim_{x\to a^-}$ denote limits as x approaches a through values that are greater than and smaller than a respectively.

5 min

Activity 1 **Basic Question**

1. Evaluate the following limits:

a) $\lim_{x\to 5}(2x-4)$ b) $\lim_{x\to 1}\frac{1}{x-3}$ c) $\lim_{x\to\frac{\pi}{4}}\tan x$

2. Show that $\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1$.

Concept 21
Application on $\lim_{x\to 0}\frac{\sin ax}{bx} = \frac{a}{b}, b \neq 0$

Example 6 Find $\lim_{x\to 0}\frac{\sin 2x}{x}$

Solution $\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{x\to 0}\left[2\left(\frac{\sin 2x}{2x}\right)\right] = \lim_{2x\to 0}\left[2\left(\frac{\sin 2x}{2x}\right)\right]$

The last equality is due to the fact that $x \to 0$ is equivalent to $2x \to 0$.

$\lim_{2x\to 0}\left[2\left(\frac{\sin 2x}{2x}\right)\right] = 2\lim_{2x\to 0}\left(\frac{\sin 2x}{2x}\right) = 2\lim_{y\to 0}\left(\frac{\sin y}{y}\right) = 2$

The last equality between limits is obtained by substituting y for $2x$.

Example 7 Find $\lim_{x\to 0}\frac{\tan 5x}{3x}$

Solution $\lim_{x\to 0}\frac{\tan 5x}{3x} = \lim_{x\to 0}\frac{\frac{\sin 5x}{\cos 5x}}{3x} = \lim_{x\to 0}\left[\left(\frac{1}{\cos 5x}\right)\left(\frac{\sin 5x}{3x}\right)\right] = \lim_{x\to 0}\left[\left(\frac{1}{\cos 5x}\right)\frac{5}{3}\left(\frac{\sin 5x}{5x}\right)\right]$

$= \frac{5}{3}\left[\lim_{x\to 0}\left(\frac{1}{\cos 5x}\right)\right]\left[\lim_{5x\to 0}\left(\frac{\sin 5x}{5x}\right)\right] = \frac{5}{3}(1)(1) = \frac{5}{3}$

5 min

Activity 2 **Basic Questions**

Find the following limits.

a) $\lim_{x\to 0}\frac{\sin 2x}{3x}$ b) $\lim_{x\to 0}\frac{\tan ax}{bx}, b \neq 0$ c) $\lim_{x\to\frac{\pi}{2}}\frac{\sin^2\left(x-\frac{\pi}{2}\right)}{\left[2\left(x-\frac{\pi}{2}\right)\right]^2}$ $\frac{1}{4}$

d) $\lim_{x\to -1}\frac{x^2+2x+1}{x^2-1}$ $\frac{(x+1)(x+1)}{(x-1)(x+1)}$ e) $\lim_{x\to 1}\frac{x^2-1}{\sqrt{x}-1}$ f) $\lim_{x\to 2}\frac{x^3-8}{x^2-4}$ $\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$ $\frac{12}{4} = 3$

A limit useful for the development of the derivative of the trigonometric functions in the next chapter is $\lim_{x\to 0}\frac{\cos x - 1}{x}$.

$\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0}\frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)}$

$= \lim_{x\to 0}\frac{\cos^2 x - 1}{x(\cos x + 1)}$

$= \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x + 1)}$

$= -\lim_{x\to 0}\left(\frac{\sin x}{x}\frac{\sin x}{\cos x + 1}\right)$

$= -\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\lim_{x\to 0}\left(\frac{\sin x}{\cos x + 1}\right) = -1\left(\frac{0}{2}\right) = 0$

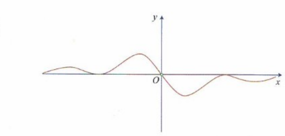

Figure 1.2.2(c) The graph of $y = \frac{\cos x - 1}{x}$