

## Inverse Function

Given a function  $f(x)$ , we call an inverse function  $f^{-1}(x)$ , when:

$$f^{-1} \circ f(x) = x \quad \text{and} \quad f \circ f^{-1}(x) = x$$

Example:  $f(x) = 3x$      $f^{-1}(x) = \frac{1}{3}x$      $\longrightarrow$      $f \circ f^{-1}(x) = f\left(\frac{1}{3}x\right) = 3\left(\frac{1}{3}x\right)$   
 $= x$

Thus these two functions are inverses.     $f^{-1} \circ f(x) = f^{-1}(3x) = \frac{1}{3}(3x)$   
 $= x$

$g(x) = \frac{1}{x}$      $g^{-1}(x) = x$      $\longrightarrow$      $g \circ g^{-1}(x) = g(x) = \frac{1}{x}$

Thus these two functions are **not** inverses.

## Finding Inverse Functions

An inverse function is a function that takes an output, and maps it back to the input. This means, to find the inverse of a function we must:

Step 1: Write the function replacing  $f(x)$  with  $y$ . (Just to make it a bit easier).

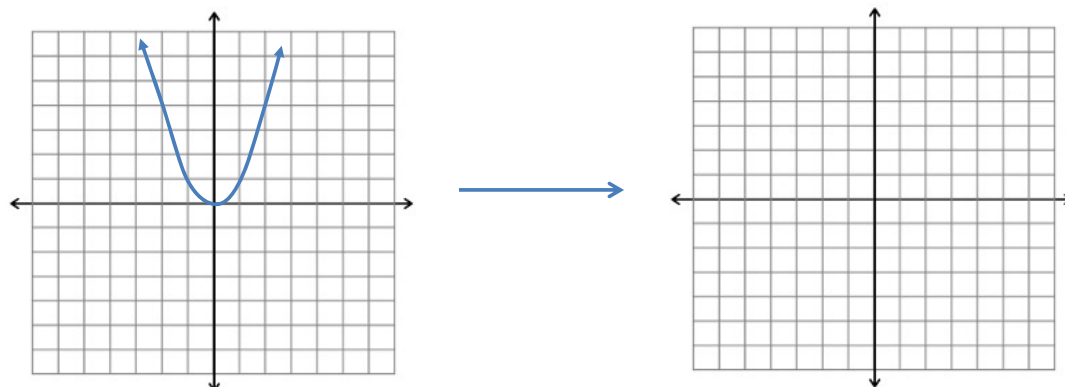
Step 2: Switch the places of  $x$  and  $y$ .

Step 3: Solve for the  $y$  in the new equation.

Step 4: We write the new function replacing  $f^{-1}(x)$  (only if it is a function)

## Graphing Inverses

If you are given a graph of a relation, you can graph the inverse of this relation by switching the x and y coordinates of each point (choose key points to help save you time).



Note that the inverse of a function may not need be a function.

How can we tell when the inverse a function is indeed a function?

## Inverse Trig Functions

Mathematicians had to invent functions to use to act as inverses to other known functions.  $\text{Arctan}(x)$  (Also called  $\text{Tan}^{-1}(x)$ ),  $\text{ArcSin}(x)$  (Also called  $\text{Sin}^{-1}(x)$ ), and  $\text{ArcCos}(x)$  (Also called  $\text{Cos}^{-1}(x)$ ) were invented to act as inverse functions to  $\text{Tan}$ ,  $\text{Sin}$ , and  $\text{Cos}$ .

$$y = \text{Tan}(x)$$

$$\text{ArcTan}(y) = x$$

$$y = \text{Sin}(x)$$

$$\text{ArcSin}(y) = x$$

$$y = \text{Cos}(x)$$

$$\text{ArcCos}(y) = x$$

Since  $f^{-1}(x) = \text{ArcTan}(x)$  is the inverse function for  $f(x) = \text{Tan}(x)$ , we get the following:

$$f^{-1} \circ f(x) = \text{ArcTan}(\text{Tan}(x)) = x$$

$$f \circ f^{-1}(x) = \text{Tan}(\text{ArcTan}(x)) = x$$

Similarly, we could say the same thing about the other two functions:

$$\text{ArcSin}(\text{Sin}(x)) = x$$

$$\text{Sin}(\text{ArcSin}(x)) = x$$

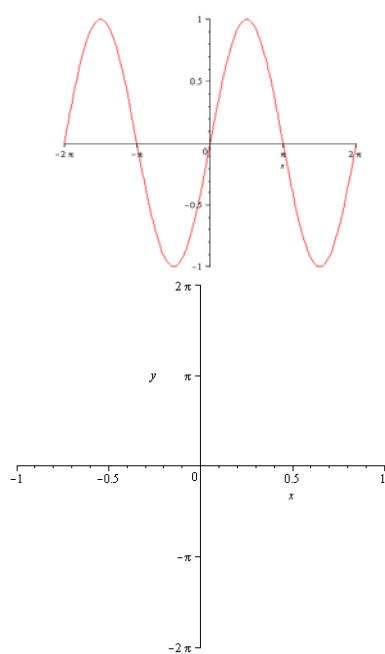
$$\text{ArcCos}(\text{Cos}(x)) = x$$

$$\text{Cos}(\text{ArcCos}(x)) = x$$

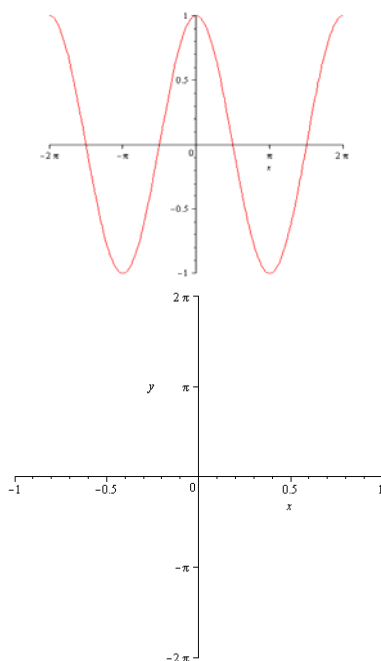
## Graphing Inverse Trig Functions

Recall, to graph an inverse function we must switch the rolls of  $x$  and  $y$  on key points on the graph. This means we get the following inverse function graphs from the base graphs.

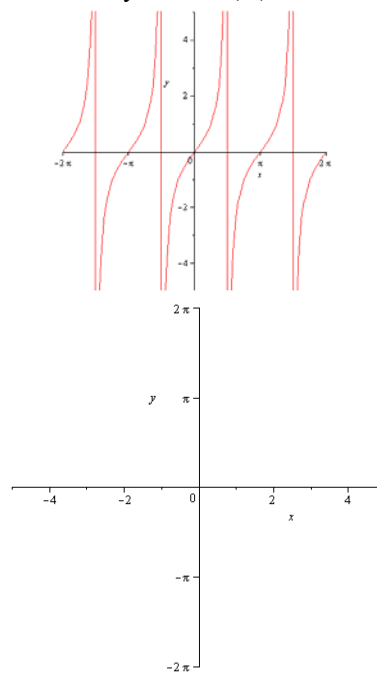
$$y = \sin(\theta)$$



$$y = \cos(\theta)$$



$$y = \tan(\theta)$$



## Solving Inverse Trig Equations

When solving inverse trig functions, we should create a new equation that has the original trig function and solve that instead:

Ex when solving:  $\text{ArcTan}(2) = x$  , we should solve:  $2 = \text{Tan}(x)$

When solving  $\text{ArcTan}(x) = 0$  , we should solve:  $x = \text{Tan}(0)$

Note that we have to keep in mind the proper range when solving for inverse trig functions:

$$\text{ArcTan}(x)=y \quad \frac{-\pi}{2} < y < \frac{\pi}{2}$$

$$\text{ArcSin}(x)=y \quad \frac{-\pi}{2} < y < \frac{\pi}{2}$$

$$\text{ArcCos}(x)=y \quad 0 < y < \pi$$

## Example 1

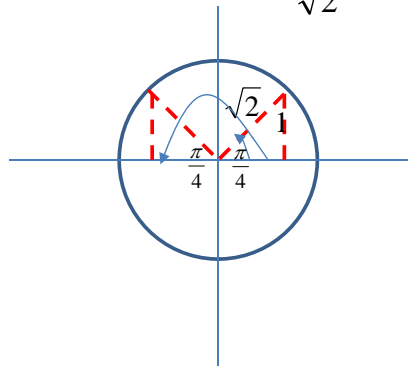
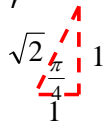
Evaluate the following:  $\text{Tan}(\text{ArcSin}(\frac{1}{\sqrt{2}}))$

We first consider solving:  $\theta = \text{ArcSin}(\frac{1}{\sqrt{2}})$

We change it into regular trig functions as:  $\text{Sin}(\theta) = \frac{1}{\sqrt{2}}$

$$\text{Sin}(\theta) = \frac{1}{\sqrt{2}} \quad \begin{matrix} y=1 \\ r=\sqrt{2} \end{matrix}$$

$$\text{Sin}(\theta) = \frac{y}{r}$$



$$\therefore \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\therefore \text{Tan}(\text{ArcSin}(\frac{1}{\sqrt{2}})) = \text{Tan}(\frac{\pi}{4}) = 1$$

$$\therefore \theta = \frac{\pi}{4}$$

## Example 2

Determine the inverse function of the following function  $f(x) = \frac{3x+2}{x-1}$

$$y = \frac{3x+2}{x-1}$$

$$x = \frac{3y+2}{y-1}$$

$$x(y-1) = 3y+2$$

$$xy - x = 3y + 2$$

$$xy - 3y = 2 + x$$

$$y(x-3) = 2+x$$

$$y = \frac{2+x}{x-3}$$

$$f^{-1}(x) = \frac{2+x}{x-3}$$

## Example 3

Determine the inverse relation of the following function  $f(x) = 2x^2 + 8x - 4$

$$y = 2x^2 + 8x - 4$$

$$x = 2y^2 + 8y - 4$$

$$x = 2(y^2 + 4y) - 4$$

$$x = 2(y^2 + 4y) - 4$$

$$x = 2(y^2 + 4y + 4 - 4) - 4$$

$$x = 2(y^2 + 4y + 4) - 8 - 4$$

$$x = 2(y^2 + 4y + 4) - 12$$

$$x = 2(y + 2)(y + 2) - 12$$

$$x = 2(y + 2)^2 - 12$$

$$x + 12 = 2(y + 2)^2$$

$$\frac{x + 12}{2} = (y + 2)^2$$

$$\left(\frac{4}{2}\right)^2 = 4$$

$$\pm \sqrt{\frac{x + 12}{2}} = y + 2$$

$$-2 \pm \sqrt{\frac{x + 12}{2}} = y$$

Note that we cannot use function notation here as this equation is not a function. We could however restrict the domain in the original function to force the inverse to be a function.