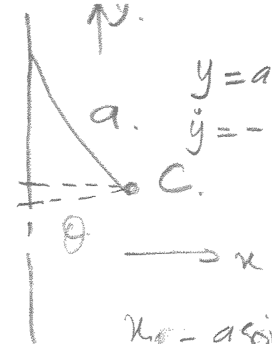
2004 Q11

$$KE = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} m V_2^2$$



$y = a\cos\theta$

$\dot{y} = -\dot{\theta} a \sin\theta$

$x_{relative} = a\sin\theta$

$\dot{x}_{relative} = \dot{\theta} a \cos\theta$

$V = R\dot{\theta}$

$$V_2^2 = \dot{x}_2^2 + \dot{y}_2^2$$

$$= V_2^2 = (\dot{x}_1 + \dot{\theta} a \cos\theta)^2 + (\dot{\theta} a \sin\theta)^2$$

$$KE = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} m \left[(\dot{x}_1 + \dot{\theta} a \cos\theta)^2 + (\dot{\theta} a \sin\theta)^2 \right]$$

$$KE = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} m \left[\dot{x}_1^2 + 2\dot{x}_1 \dot{\theta} a \cos\theta + (\dot{\theta} a \cos\theta)^2 + (\dot{\theta} a \sin\theta)^2 \right]$$

$$KE = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} m \dot{x}_1^2 + \frac{2}{2} m \dot{x}_1 \dot{\theta} a \cos\theta + \frac{1}{2} m \dot{\theta}^2 a^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}$$

$$KE = \frac{1}{2}(M+m)\dot{x}_1^2 + \frac{2}{2} m \dot{x} \dot{\theta} a \cos\theta + \frac{1}{2} m a^2 \dot{\theta}^2$$

$$PE = \frac{1}{2} K x_1^2 + mga(1 - \cos\theta)$$

$$D = \frac{1}{2} c \dot{x}_1^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} + \frac{\partial D}{\partial \dot{x}} = f$$

$$L = KE - PE = \frac{1}{2}(M+m)\dot{x}_1^2 + m \dot{x}_1 \dot{\theta} a \cos\theta + \frac{1}{2} m a^2 \dot{\theta}^2 - \frac{1}{2} K x_1^2 - mga(1-\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{d}{dt}\left[(M+m)\dot{x}_1 + m\dot{\theta} a \cos\theta\right] = (M+m)\ddot{x}_1 + ma\cos\theta\,\ddot{\theta} - m\dot{\theta}^2 a \sin\theta$$

$$\frac{\partial L}{\partial x_1} = -K x_1 \quad ; \quad \frac{\partial D}{\partial \dot{x}_1} = c\dot{x}_1$$

$$(M+m)\ddot{x}_1 + ma\cos\theta\,\ddot{\theta} - ma\sin\theta\,\dot{\theta}^2 + c\dot{x}_1 + Kx_1 = f_1$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\theta}}\right) + \frac{\delta D}{\delta \dot{\theta}} - \frac{\delta L}{\delta \theta} = f_2 .$$

$$\left(T = f_2 \, a \cos\theta \right)$$

$$\left[f_2 = \frac{T}{a\cos\theta} \right].$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\theta}}\right) = \frac{d}{dt}\left[ma\cos\theta\,\dot{x}_1 + ma^2\dot{\theta}\right] = ma\cos\theta\,\ddot{x}_1 - ma\sin\theta\,\dot{\theta}\dot{x}_1 + ma^2\ddot{\theta}$$

$$\frac{\delta D}{\delta \dot{\theta}} = 0$$

$$\frac{\delta L}{\delta \theta} = -ma\dot{x}\dot{\theta}\sin\theta - mga\sin\theta$$

$$ma\cos\theta\,\ddot{x}_1 - \cancel{ma\sin\theta\,\dot{\theta}\dot{x}_1} + ma^2\ddot{\theta} + \cancel{ma\dot{x}\dot{\theta}\sin\theta} + mga\sin\theta = f_2 .$$

$$ma\cos\theta\,\ddot{x}_1 + ma^2\ddot{\theta} + mga\sin\theta = f_2$$

c) linearized equation of motion.

$$\cos\theta \approx 1 \quad ; \quad \sin\theta \simeq \theta .$$

$$(M+m)\ddot{x}_1 + ma\ddot{\theta} + C\dot{x}_1 + Kx_1 = f_1$$

$$ma\ddot{x}_1 + ma^2\ddot{\theta} + mga\,\theta = f_2 .$$

d) $$\begin{bmatrix} M+m & ma \\ ma & ma^2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & mga \end{bmatrix}\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Prob 9

Q3 $\quad \ddot{x} + 2\zeta\omega \dot{x} + \omega_n^2 x = F(t)$

a) find the frequency response $T(i\omega)$.

$F(t) = F_0 e^{i\omega t}$; $x(t) =$

Laplace $\rightarrow$ $[s^2 + 2\zeta\omega_n s + \omega_n^2] X(s) = Y(s)$. (IC = 0)

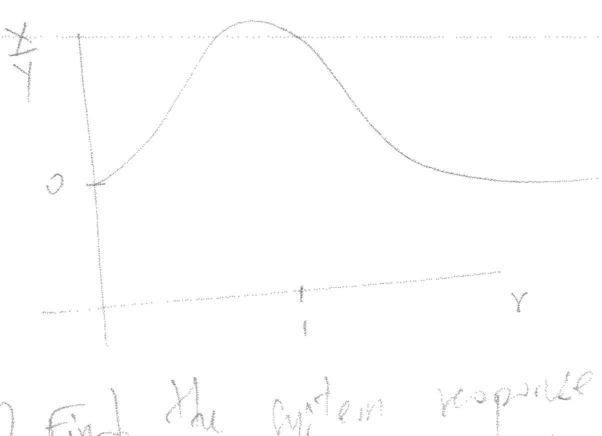
$$H(s) = \frac{X(s)}{Y(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (s = i\omega)$$

$$T(i\omega) = \frac{1}{-\omega^2 + i2\zeta\omega_n\omega + \omega_n^2} = \frac{1}{(\omega_n^2 - \omega^2) + i(2\zeta\omega_n\omega)}$$

$$M = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

$$\phi = 0 - \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

b)



c) Find the system response to the sawtooth function.

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{-1}{n\pi}\right) \sin(nt).$$

$T = 2\pi \qquad \omega_1 = \frac{2\pi}{T} = 1$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{2}{2\pi}\int_0^{2\pi} \frac{1}{2\pi} t\,dt \qquad \left(f(t) = \frac{1}{2\pi} t \quad 0 < t < 2\pi\right)$$

$$a_0 = \frac{1}{2\pi^2}\int_0^{2\pi} t\,dt = \frac{1}{2\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi^2}\left[\frac{4\pi^2}{2}\right] = 1.$$

$$a_n = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\cos(n\omega t)\,dt = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2\pi}\, t\cos(n\omega t)\,dt$$

$$a_n = \frac{1}{2\pi^2}\int_0^{2\pi} t\cos(n\omega t)\,dt$$

$$u = t\ ,\ v' = \cos(n\omega t) \Rightarrow u' = 1\ ;\ v = \frac{1}{n\omega}\sin(n\omega t)$$

$$\int u\,dv = uv - \int v\,du = \frac{t}{n\omega}\sin(n\omega t) - \int_0^{2\pi}\left(\frac{1}{n\omega}\right)\sin(n\omega t)\,.\,dt$$

$$a_n = \frac{1}{2\pi^2}\left[\left[\frac{t}{n\omega}\sin(n\omega t)\right]_0^{2\pi} - \int_0^{2\pi}\frac{1}{n\omega}\sin(n\omega t)\,dt\right] = 0$$

$$b_n = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\sin(n\omega t)\,dt = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2\pi}\,t\,.\sin(n\omega t)\,dt.$$

$$u = t\ ,\ v' = \sin(n\omega t) \Rightarrow u' = 1\ ;\ v = -\frac{1}{n\omega}\cos(n\omega t)$$

$$\int u\,dv = uv - \int v\,du = -\frac{t}{n\omega}\cos(n\omega t) + \int \frac{1}{n\omega}\cos(n\omega t)\,dt$$

$$b_n = \frac{1}{2\pi^2}\left[\left[-\frac{t}{n\omega}\cos(n\omega t)\right]_0^{2\pi} + \int_0^{2\pi}\frac{1}{(n\omega)^2}\overset{0}{\cancel{\sin(n\omega t)}}\,dt\right] \quad (\omega = 1)$$

$$b_n = -\frac{1}{2\pi^2 n\omega}\left[+2\pi\cos(2\pi n\omega) - 0\right] = -\frac{1}{\pi n}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(n\omega t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{1}{\pi n}\right)\sin(n\omega t)$$

d) Yes, $a_{0_C} = 1 = a_{0_D}$ $\left(F(t)_D = f(t)_C\right)$.

2009 Q4

$$\overset{M}{\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}} \ddot{\vec{x}} + \overset{K}{\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}} \vec{x} = \vec{0}$$

a) Find the natural frequencies and mode shapes.

$$\det(\lambda^2 M + K) = 0.$$

$$\det\left(\begin{bmatrix} 2\lambda^2+3 & \lambda^2 \\ \lambda^2 & 5\lambda^2+6 \end{bmatrix}\right) = 0 \rightarrow (2\lambda^2+3)(5\lambda^2+6) - \lambda^4 = 0.$$

$$10\lambda^4 + 12\lambda^2 + 15\lambda^2 + 18 - \lambda^4 = 0$$

$$9\lambda^4 + 27\lambda^2 + 18 = 0 \Rightarrow \lambda^4 + 3\lambda^2 + 2 = 0$$

Charact. eq.

$$\lambda_{13}^2 = -1 \qquad \lambda_{24}^2 = -2.$$

$$\lambda_{13} = \pm i\omega_1 \rightarrow \omega_1 = 1 \quad ; \quad \lambda_{24} = \pm i\omega_2 \rightarrow \omega_2 = \sqrt{2}.$$

$$(\lambda^2 M + K)\vec{u} = 0.$$

$$\begin{bmatrix} 2\lambda^2+3 & \lambda^2 \\ \lambda^2 & 5\lambda^2+6 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_{13}^2 \rightarrow \begin{bmatrix} -2+3 & -1 \\ -1 & -5+6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow u_{11} - u_{12} = 0$ and $-u_{11} + u_{12} = 0$.

Let set $u_{12} = 1 \rightarrow u_{11} = 1 \rightarrow u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

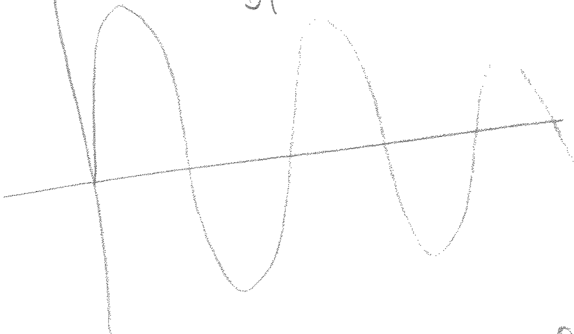
$$\lambda_{24} \rightarrow \begin{bmatrix} -4+3 & -2 \\ -2 & -10+6 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow -u_{21} - 2u_{22} = 0$ and $-2u_{21} - 4u_{22} = 0$.

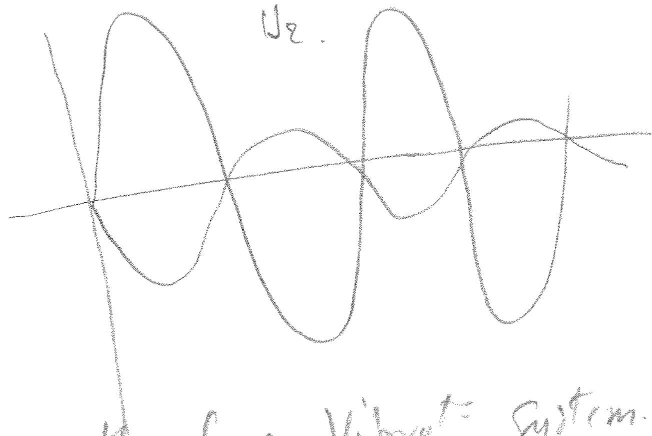
Let set $u_{22} = 1 \rightarrow u_{21} = -2. \rightarrow u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

b) Sketch the mode shape.

U_1



U_2.



c) Write the general form of the free Vibrat[ion] System.

$$x(t) = a e^{i\omega_1 t} u_1 + b e^{-i\omega_1 t} u_1 + c e^{i\omega_2 t} u_2 + d e^{-i\omega_2 t} u_2.$$

d) The equation of motion in modal coordinates

$$M_{new} = P^T M P. \qquad P = [u_1 \ u_2].$$

$$K_{new} = P^T K P.$$

$$\vec{U_1} = \frac{1}{\sqrt{m_{11}}} \vec{u_1} \qquad \vec{U_2} = \frac{1}{\sqrt{m_{22}}} \vec{u_2} \rightarrow P_n = [U_1 \ U_2].$$

$$M_{new} \ddot{\vec{q}} + K_{new} \vec{q} = \vec{Q} \qquad (\vec{x} = P \vec{q}).$$

e) Find the mass-normalized modal matrix.

$$M_{new} = P_n^T M P_n \ ; \ \vec{U_1} = \frac{1}{\sqrt{m_{11}}} \vec{u_1} \ ; \ m_{11} = u_1^T M . u_2$$

$$m_{11} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 + 6 = 9 \rightarrow \vec{U_1} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

$$m_{22} = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2 + 3 = 5 \rightarrow \vec{U_2} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$P_n = \begin{bmatrix} 1/3 & -2/\sqrt{5} \\ 1/3 & 1/\sqrt{5} \end{bmatrix} \rightarrow M_{new} = \begin{bmatrix} 1/3 & 1/3 \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1/3 & -2/\sqrt{5} \\ 1/3 & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3/\sqrt{5} & 4/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/3 & -2/\sqrt{5} \\ 1/3 & 1/\sqrt{5} \end{bmatrix}$$

$$M_{new} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

2009 Q5

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \ddot{\vec{x}} + \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \dot{\vec{x}} + \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} -2\cos(t) \\ 2\cos(t) \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

a) Yes. $C = \alpha M + \beta K$. (with $\alpha = 0$ and $\beta = 1$).

b) Using the modal matrix find the EoM in modal coordinates.

$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = F$ ($x = P \cdot q$).

$MP\ddot{\vec{q}} + CP\dot{\vec{q}} + KP\vec{q} = F$. (multiply by P^T). (since C is proportion

$P^T MP\ddot{\vec{q}} + P^T CP\dot{\vec{q}} + P^T KP\vec{q} = P^T F$.

$M_{new}\ddot{\vec{q}} + C_{new}\dot{\vec{q}} + K_{new}\vec{q} = \vec{Q}$.

$$M_{new} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$C_{new} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix}$$

$$K_{new} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix}$$

$$Q = P^T F = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2\cos t \\ 2\cos t \end{bmatrix} = \begin{bmatrix} 0 \\ 4\cos t \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\vec{q}} + \begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix}\dot{\vec{q}} + \begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix}\vec{q} = \begin{bmatrix} 0 \\ 4\cos t \end{bmatrix}.$$

c) Mode 1: $\omega_{n1} = \sqrt{\frac{k}{m}} = 1$. $\omega d_1 = \omega_{n1}\sqrt{1-\zeta^2}$. $\zeta = \frac{c}{2m\omega_n} = \frac{4}{2\times4\times1} = 0{,}5$

$\omega d_1 = \frac{1}{\sqrt{2}} = 0{,}86$

Mode 2: $\omega_n = \sqrt{\frac{12}{4}} = \sqrt{3}$ $\omega d_2 = \omega_{n2}\sqrt{1-\zeta^2}$ $\zeta = \frac{12}{2\times4\times\sqrt{3}} = 0{,}86$

d) No, $\omega n \neq \omega d$ $\omega d_2 = 0{,}88$