

CVG2140 – Solutions to Assignment No. 6 (Deflections)

Problem 1. Determine the equations of the elastic curve for the beam illustrated in Fig. 1 using the x coordinate as shown. Specify the slope at A and the maximum displacement of the shaft. EI is constant.

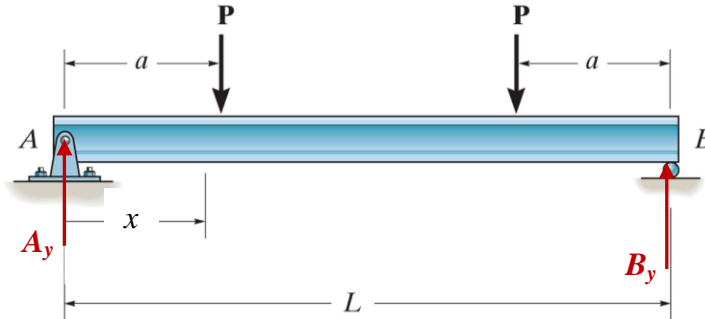


Fig. 1

Establishing equilibrium:

$$\left. \begin{aligned} \sum F_y &= A_y - 2P + B_y = 0 \\ \sum M_A &= -(P \times a) - [P \times (L - a)] + (B_y \times L) = 0 \end{aligned} \right\} \Rightarrow \begin{cases} A_y = P \\ B_y = P \end{cases}$$

Using discontinuity functions, the moment equation is written as:

$$M(x) = P\langle x - 0 \rangle - P\langle x - a \rangle - P\langle x - (L - a) \rangle$$

$$EI \frac{d^2v}{dx^2} = M(x) = Px - P\langle x - a \rangle - P\langle x - (L - a) \rangle$$

$$EI \frac{dv}{dx} = \frac{P}{2}x^2 - \frac{P}{2}\langle x - a \rangle^2 - \frac{P}{2}\langle x - (L - a) \rangle^2 + C_1 \quad (\text{Eq. 1})$$

$$EIv = \frac{P}{6}x^3 - \frac{P}{6}\langle x - a \rangle^3 - \frac{P}{6}\langle x - (L - a) \rangle^3 + C_1x + C_2 \quad (\text{Eq. 2})$$

The boundary conditions of the problem are:

$$x = 0 \Rightarrow v = 0$$

$$x = L \Rightarrow v = 0$$

The constants of integration C_1 and C_2 are then determined by applying the boundary conditions to Eq. 2:

$$x = 0 \Rightarrow v = 0 = C_2$$

$$x = L \Rightarrow v = 0 = \frac{P}{6}L^3 - \frac{P}{6}(L - a)^3 - \frac{P}{6}a^3 + C_1L \Rightarrow C_1 = -\frac{Pa}{2}(L - a)$$

The equations for the slope and deflection are therefore given by:

$$EI \frac{dv}{dx} = \frac{P}{2}x^2 - \frac{P}{2}\langle x - a \rangle^2 - \frac{P}{2}\langle x - (L - a) \rangle^2 - \frac{Pa}{2}(L - a) \quad (\text{Eq. 3})$$

$$EIv = \frac{P}{6}x^3 - \frac{P}{6}\langle x - a \rangle^3 - \frac{P}{6}\langle x - (L - a) \rangle^3 - \frac{Pax}{2}(L - a) \quad (\text{Eq. 4})$$

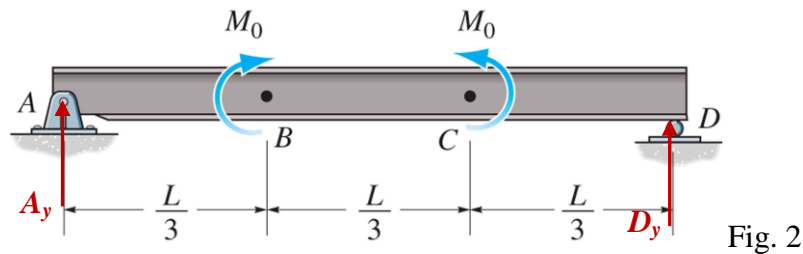
The slope at A is calculated from Eq. 3 for $x = 0$:

$$\theta_A = \frac{1}{EI} \left[\frac{P}{2}(0)^2 - 0 - 0 - \frac{Pa}{2}(L-a) \right] = -\frac{Pa}{2EI}(L-a) \quad (\text{rotation is clockwise})$$

Because of symmetry in loading and boundary conditions, the maximum deflection occurs at $x = L/2$. Using Eq. 4:

$$v_{\max} = \frac{1}{EI} \left[\frac{P}{6} \left(\frac{L}{2} \right)^3 - \frac{P}{6} \left(\frac{L}{2} - a \right)^3 - 0 - \frac{PaL}{4}(L-a) \right] = \frac{Pa}{24EI} (4a^2 - 3L^2)$$

Problem 2. Determine the equation of the elastic curve, the slope at A, and the deflection at B of the simply supported beam shown in Fig. 2. EI is constant.



Establishing equilibrium:

$$\left. \begin{aligned} \sum F_y = A_y + D_y = 0 \\ \sum M_A = -M_o + M_o + (D_y \times L) = 0 \end{aligned} \right\} \Rightarrow \begin{cases} A_y = 0 \\ D_y = 0 \end{cases}$$

Using discontinuity functions, the moment equation is written as:

$$M(x) = M_o \left\langle x - \frac{L}{3} \right\rangle^0 - M_o \left\langle x - \frac{2L}{3} \right\rangle^0$$

$$EI \frac{d^2v}{dx^2} = M(x) = M_o \left\langle x - \frac{L}{3} \right\rangle^0 - M_o \left\langle x - \frac{2L}{3} \right\rangle^0$$

$$EI \frac{dv}{dx} = M_o \left\langle x - \frac{L}{3} \right\rangle - M_o \left\langle x - \frac{2L}{3} \right\rangle + C_1 \quad (\text{Eq. 1})$$

$$EIv = \frac{M_o}{2} \left\langle x - \frac{L}{3} \right\rangle^2 - \frac{M_o}{2} \left\langle x - \frac{2L}{3} \right\rangle^2 + C_1x + C_2 \quad (\text{Eq. 2})$$

The boundary conditions of the problem are:

$$x = 0 \Rightarrow v = 0$$

$$x = L \Rightarrow v = 0$$

The constants of integration C_1 and C_2 are then determined by applying the boundary conditions to Eq. 2:

$$x = 0 \Rightarrow v = 0 = C_2$$

$$x = L \Rightarrow v = 0 = \frac{M_o}{2} \left(\frac{2L}{3} \right)^2 - \frac{M_o}{2} \left(\frac{L}{3} \right)^2 + C_1L \Rightarrow C_1 = -\frac{M_oL}{6}$$

The equations for the slope and deflection are therefore given by:

$$EI \frac{dv}{dx} = M_o \left\langle x - \frac{L}{3} \right\rangle - M_o \left\langle x - \frac{2L}{3} \right\rangle - \frac{M_o L}{6} \quad (\text{Eq. 3})$$

$$EIv = \frac{M_o}{2} \left\langle x - \frac{L}{3} \right\rangle^2 - \frac{M_o}{2} \left\langle x - \frac{2L}{3} \right\rangle^2 - \frac{M_o L}{6} x \quad (\text{Eq. 4})$$

The slope at A is calculated from Eq. 3 for $x = 0$:

$$\theta_A = -\frac{M_o L}{6EI}$$

The deflection at B is calculated from Eq. 4 for $x = L/3$:

$$v_B = \frac{1}{EI} \left(-\frac{M_o L}{6} \cdot \frac{L}{3} \right) = -\frac{M_o L^2}{18EI}$$

Problem 3. Determine the maximum deflection of the simply supported beam shown in Fig. 3. $E = 200 \text{ GPa}$ and $I = 65.0 \times 10^6 \text{ mm}^4$.

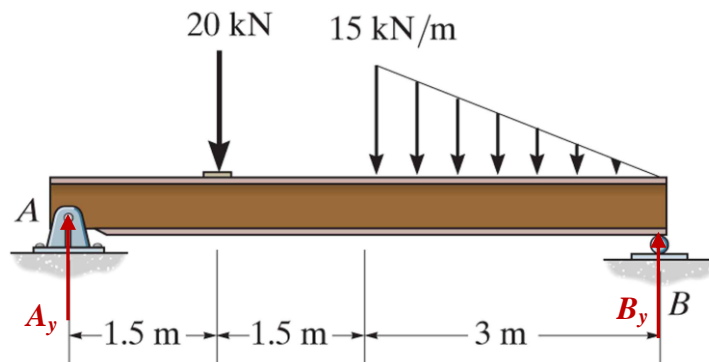


Fig. 3

Establishing equilibrium:

$$\left. \begin{aligned} \sum F_y &= A_y - 20 - \left(\frac{1}{2} \times 15 \times 3 \right) + B_y = 0 \\ \sum M_A &= -(20 \times 1.5) - \left(\frac{1}{2} \times 15 \times 3 \right) (3+1) + (B_y \times 6) = 0 \end{aligned} \right\} \Rightarrow \begin{cases} A_y = 22.5 \text{ kN} \\ B_y = 20 \text{ kN} \end{cases}$$

Using discontinuity functions, the moment equation is written as:

$$M(x) = 22.5 \langle x-0 \rangle - 20 \langle x-1.5 \rangle - \frac{15}{2} \langle x-3 \rangle^2 + \frac{15}{18} \langle x-3 \rangle^3 - \frac{15}{18} \langle x-6 \rangle^3$$

$$EI \frac{d^2v}{dx^2} = M(x) = 22.5x - 20 \langle x-1.5 \rangle - \frac{15}{2} \langle x-3 \rangle^2 + \frac{15}{18} \langle x-3 \rangle^3 - \frac{15}{18} \langle x-6 \rangle^3$$

$$EI \frac{dv}{dx} = \frac{22.5}{2} x^2 - \frac{20}{2} \langle x-1.5 \rangle^2 - \frac{15}{6} \langle x-3 \rangle^3 + \frac{15}{72} \langle x-3 \rangle^4 - \frac{15}{72} \langle x-6 \rangle^4 + C_1 \quad (\text{Eq. 1})$$

$$EIv = \frac{22.5}{6} x^3 - \frac{20}{6} \langle x-1.5 \rangle^3 - \frac{15}{24} \langle x-3 \rangle^4 + \frac{15}{360} \langle x-3 \rangle^5 - \frac{15}{360} \langle x-6 \rangle^5 + C_1 x + C_2 \quad (\text{Eq. 2})$$

The boundary conditions of the problem are:

$$x = 0 \Rightarrow v = 0$$

$$x = 6 \Rightarrow v = 0$$

The constants of integration C_1 and C_2 are then determined by applying the boundary conditions to Eq. 2:

$$x = 0 \Rightarrow v = 0 = C_2$$

$$x = 6 \Rightarrow v = 0 = (22.5 \times 6^2) - \left(\frac{20}{6} \times 4.5^3\right) - \left(\frac{15}{24} \times 3^4\right) + \left(\frac{15}{360} \times 3^5\right) + 6C_1 \Rightarrow$$

$$C_1 = -77.625$$

The equations for the slope and deflection are therefore given by:

$$EI \frac{dv}{dx} = 11.25x^2 - 10\langle x-1.5 \rangle^2 - 2.5\langle x-3 \rangle^3 + 0.2083\langle x-3 \rangle^4 - 0.2083\langle x-6 \rangle^4 - 77.625 \quad (\text{Eq. 3})$$

$$EIv = 3.75x^3 - 3.33\langle x-1.5 \rangle^3 - 0.625\langle x-3 \rangle^4 + 0.0417\langle x-3 \rangle^5 - 0.0417\langle x-6 \rangle^5 - 77.625x \quad (\text{Eq. 4})$$

The maximum deflection occurs when the slope is zero. Assuming that $\frac{dv}{dx} = 0$ occurs in the region $1.5 \text{ m} < x < 3 \text{ m}$, then:

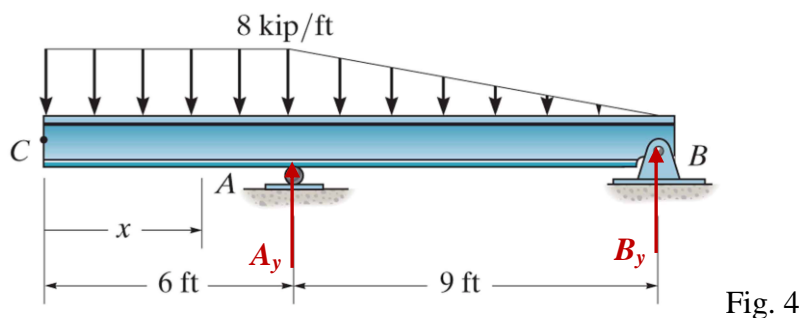
$$\frac{dv}{dx} = 0 = \frac{1}{EI} [11.25x^2 - 10(x-1.5)^2 - 0 + 0 - 0 - 77.625]$$

Solving for the root $1.5 \text{ m} < x < 3 \text{ m}$, $x = 2.97 \text{ m}$ (the solution is within assumed bounds, therefore it is O.K.)

The maximum deflection is calculated from Eq. 4 for $x = 2.97 \text{ m}$:

$$v_{\max} = \frac{1}{200 \times 10^6 \times 65 \times 10^{-6}} [3.75(2.97)^3 - 3.33(1.47)^3 - 77.625(2.97)] = \underline{-0.01099 \text{ m}}$$

Problem 4. Determine the displacement at C and the slope at A of the beam shown in Fig. 4.



Establishing equilibrium:

$$\left. \begin{aligned} \sum F_y &= A_y - (8 \times 6) - \left(\frac{1}{2} \times 8 \times 9\right) + B_y = 0 \\ \sum M_A &= (8 \times 6 \times 3) - \left(\frac{1}{2} \times 8 \times 9\right)(3) + (B_y \times 9) = 0 \end{aligned} \right\} \Rightarrow \begin{cases} A_y = 88 \text{ kip} \\ B_y = -4 \text{ kip} \end{cases}$$

Using discontinuity functions, the moment equation is written as:

$$M(x) = -\frac{8}{2}\langle x-0 \rangle^2 + \frac{8}{2}\langle x-6 \rangle^2 + 88\langle x-6 \rangle - \frac{8}{2}\langle x-6 \rangle^2 + \frac{8}{6 \times 9}\langle x-6 \rangle^3 - \frac{8}{6 \times 9}\langle x-15 \rangle^3 - 4\langle x-15 \rangle$$

$$EI \frac{d^2v}{dx^2} = M(x) = -4x^2 + 88\langle x-6 \rangle + \frac{4}{27}\langle x-6 \rangle^3$$

$$EI \frac{dv}{dx} = -\frac{4}{3}x^3 + \frac{88}{2}\langle x-6 \rangle^2 + \frac{4}{27 \times 4}\langle x-6 \rangle^4 + C_1 \quad (\text{Eq. 1})$$

$$EIv = -\frac{4}{12}x^4 + \frac{88}{6}\langle x-6 \rangle^3 + \frac{4}{27 \times 20}\langle x-6 \rangle^5 + C_1x + C_2 \quad (\text{Eq. 2})$$

The boundary conditions of the problem are:

$$x = 6 \Rightarrow v = 0$$

$$x = 15 \Rightarrow v = 0$$

The constants of integration C_1 and C_2 are then determined by applying the boundary conditions to Eq. 2:

$$\left. \begin{aligned} x = 6 &\Rightarrow v = 0 = -\frac{4}{12}6^4 + 6C_1 + C_2 \\ x = 15 &\Rightarrow v = 0 = -\frac{4}{12}15^4 + \frac{88}{6}9^3 + \frac{4}{27 \times 20}9^5 + 15C_1 + C_2 \end{aligned} \right\} \Rightarrow \begin{cases} C_1 = 590.4 \\ C_2 = -3110.4 \end{cases}$$

The equations for the slope and deflection are therefore given by:

$$EI \frac{dv}{dx} = -\frac{4}{3}x^3 + \frac{88}{2}\langle x-6 \rangle^2 + \frac{4}{27 \times 4}\langle x-6 \rangle^4 + 590.4 \quad (\text{Eq. 3})$$

$$EIv = -\frac{4}{12}x^4 + \frac{88}{6}\langle x-6 \rangle^3 + \frac{4}{27 \times 20}\langle x-6 \rangle^5 + 590.4x - 3310.4 \quad (\text{Eq. 4})$$

The deflection at C is calculated from Eq. 4 for $x = 0$:

$$v_C = \frac{1}{EI} \left[-\frac{4}{12}(0)^4 + 0 + 0 + 590.4(0) - 3310.4 \right] = -\frac{3310.4 \text{ kip} \cdot \text{ft}^3}{EI}$$

The slope at A is calculated from Eq. 3 for $x = 6$:

$$\theta_A = \frac{1}{EI} \left[-\frac{4}{3}6^3 + 0 + 0 + 590.4 \right] = \frac{302.4 \text{ kip} \cdot \text{ft}^2}{EI}$$