THE UNIVERSITY OF BRITISH COLUMBIA

MATH 317

Practice Midterm 1

21 July 2015

TIME: 75 MINUTES

LAST NAME: Solution	FIRST NAME:
STUDENT # :	SIGNATURE:

This Examination paper consists of 8 pages (including this one). Make sure you have all 8.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.

MARKING:

Q1	/14
Q2	/7
Q3	/5
Q4	/12
Q5	/12
TOTAL	/45

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Q1 [14 marks = 2+3+3+3+3]

A particle is moving through space. Measurement equipment shows that at time t_0, the particle's acceleration is $\langle 2, 1, 1 \rangle$ and its velocity is $\langle 3, 4, 0 \rangle$. Find the following quantities associated with the particle's motion at time t_0:

(a) the unit tangent vector: $\boxed{\left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle}$

(b) the tangential component of acceleration: $\boxed{2}$

(c) the principal normal vector: $\boxed{\left\langle \frac{4}{5\sqrt{2}}, \frac{-3}{5\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}$

(d) the normal component of acceleration: $\boxed{\sqrt{2}}$

(e) the curvature: $\boxed{\frac{\sqrt{2}}{25}}$

(Hint: Express $\vec{\mathbf{a}}$ as a linear combination of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$. What is $\vec{\mathbf{a}} \cdot \vec{\mathbf{T}}$?)

(a) $$\vec{T}(t_0) = \frac{\vec{r}\,'(t_0)}{|\vec{r}\,'(t_0)|} = \frac{\langle 3,4,0\rangle}{|\langle 3,4,0\rangle|} = \frac{\langle 3,4,0\rangle}{\sqrt{25}} = \left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle$$

(b) $$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

$$\vec{a}\cdot\vec{T} = a_T \underbrace{\vec{T}\cdot\vec{T}}_{=1} + a_N \underbrace{\vec{N}\cdot\vec{T}}_{=0} = a_T$$

$$a_T \underset{\uparrow \text{ time } t_0}{=} \langle 2,1,1\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle = \frac{6}{5} + \frac{4}{5} = 2$$

Extra space for work.

(c) and (d): $a_N \vec{N} = \vec{a} - a_T \vec{T}$

$$\overset{\text{at time } t_0}{=} \langle 2,1,1\rangle - 2\langle \tfrac{3}{5}, \tfrac{4}{5}, 0\rangle$$

$$= \langle 2-\tfrac{6}{5}, 1-\tfrac{8}{5}, 1\rangle = \langle \tfrac{4}{5}, -\tfrac{3}{5}, 1\rangle$$

$$a_N = |a_N \vec{N}| \overset{\text{time } t_0}{=} \left|\langle \tfrac{4}{5}, -\tfrac{3}{5}, 1\rangle\right| = \left[\frac{16+9+25}{25}\right]^{1/2} = \sqrt{2}$$

$$\vec{N} = \frac{a_N \vec{N}}{a_N} \overset{\text{time } t_0}{=} \frac{\langle \tfrac{4}{5}, -\tfrac{3}{5}, 1\rangle}{\sqrt{2}} = \left\langle \frac{4}{5\sqrt{2}}, -\frac{3}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

(e): $K v^2 = a_N \implies K = \dfrac{a_N}{v^2} = \dfrac{a_N}{|\vec{r}\,'|^2}$

$$K(t_0) = \frac{a_N(t_0)}{|\vec{r}\,'(t_0)|^2} = \frac{\sqrt{2}}{(3^2+4^2+0)} = \frac{\sqrt{2}}{25}$$

Or: $K = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} \overset{\text{time } t_0}{=} \dfrac{|\langle 3,4,0\rangle \times \langle 2,1,1\rangle|}{|\langle 3,4,0\rangle|^3}$

$$= \frac{\left|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ 2 & 1 & 1 \end{vmatrix}\right|}{\sqrt{25}^{\,3}} = \frac{|\langle 4, -3, -5\rangle|}{5^3} = \frac{\sqrt{50}}{5^3} = \frac{\sqrt{2}}{25}$$

(also, $a_N = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|} = \ldots$)

Q2 [7 marks]

Find the length of the segment of the curve $\vec{\mathbf{r}}(t) = \left\langle \sqrt{t}, t, \frac{1}{2}\ln t \right\rangle$ starting at $(1,1,0)$ and ending at $(2,4,\ln 2)$.

$$\vec{r}(1) = \langle 1,1,0\rangle$$
$$\vec{r}(4) = \langle 2,4,\ln 2\rangle$$

$$\vec{r}\,'(t) = \left\langle \frac{1}{2\sqrt{t}}, 1, \frac{1}{2t} \right\rangle$$

$$|\vec{r}\,'(t)| = \left[\frac{1}{4t} + 1 + \frac{1}{4t^2}\right]^{1/2}$$

we only need $1 \le t \le 4$

$$= \left[\left(\frac{1}{2t}+1\right)^2\right]^{1/2} = \left|\frac{1}{2t}+1\right| = \frac{1}{2t}+1$$

$$L = \int_1^4 \left(\frac{1}{2t}+1\right) dt = \left[\frac{1}{2}\ln t + t\right]_1^4$$

$$= \left(\frac{1}{2}\ln 4 + 4\right) - (0+1)$$

$$= \boxed{\ln 2 + 3}$$

Q3 [5 marks]

Let $\vec{\mathbf{r}}(t)$ be a parameterized curve such that $|\vec{\mathbf{r}}'(t)| = 2$ for all t. Simplify

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t))$$

as much as possible. Your answer should not have any derivatives except $\vec{\mathbf{r}}'$, $\vec{\mathbf{r}}''$, $\vec{\mathbf{r}}'''$, etcetera.

$$\frac{d^2}{dt^2}(\vec{r} \cdot \vec{r}\,') = \frac{d}{dt}(\vec{r}\,' \cdot \vec{r}\,' + \vec{r} \cdot \vec{r}\,'')$$

$$= \frac{d}{dt}(|\vec{r}\,'|^2 + \vec{r} \cdot \vec{r}\,'')$$

$$= \frac{d}{dt}(4 + \vec{r} \cdot \vec{r}\,'')$$

$$= 0 + \vec{r}\,' \cdot \vec{r}\,'' + \vec{r} \cdot \vec{r}\,'''$$

$$= \boxed{\vec{r} \cdot \vec{r}\,'''}$$

We have $\vec{r}\,' \cdot \vec{r}\,' = |\vec{r}\,'|^2 = 4$

so $\frac{d}{dt}(\vec{r}\,' \cdot \vec{r}\,') = 0$

$\quad\| $

$2\vec{r}\,' \cdot \vec{r}\,''$

$\Rightarrow \vec{r}\,' \cdot \vec{r}\,'' = 0$

Q4 [12 marks = 6+6]

Determine whether the following vector fields are conservative. If they are conservative, find a potential function. If not, explain why they are not conservative.

(a) $\vec{F}(x,y) = (x^3 + 2xy^2 - y^3)\vec{\imath} + (2x^2y - 3xy^2 + 2y^3)\vec{\jmath}$

Try to find f with $\vec{F} = \vec{\nabla} f$

$$f_x = x^3 + 2xy^2 - y^3$$

$$f = \int (x^3 + 2xy^2 - y^3)\,dx + C(y) = \frac{x^4}{4} + x^2y^2 - xy^3 + C(y)$$

$$2x^2y - 3xy^2 + 2y^3 = f_y \stackrel{?}{=} 2x^2y - 3xy^2 + C'(y)$$

$$C'(y) = 2y^3$$

$$C(y) = \frac{1}{2}y^4 + D \quad (\text{take } D=0)$$

$$\boxed{f(x,y) = \frac{x^4}{4} + x^2y^2 - xy^3 + \frac{1}{2}y^4, \quad \vec{F} \text{ is conservative!}}$$

(b) $\vec{F}(x,y) = (xy\cos y)\vec{\imath} + (-\frac{1}{2}x^2y\sin y - \frac{1}{2}x^2\cos y)\vec{\jmath}$

Suppose $\vec{F} = \vec{\nabla} f$:

$$f_x = xy\cos y$$

$$f = \int xy\cos y\,dx + C(y) = \frac{1}{2}x^2y\cos y + C(y)$$

$$-\frac{1}{2}x^2y\sin y - \frac{1}{2}x^2\cos y = f_y \stackrel{?}{=} \frac{1}{2}x^2\cos y - \frac{1}{2}x^2y\sin y + C'(y)$$

$$C'(y) = -x^2\cos y \implies \text{impossible!}$$

$\implies$ There is no f with $F = \vec{\nabla} f$

$$\boxed{\vec{F} \text{ is not conservative!}}$$

Q5 [12 marks = 4+8]

Let C be the intersection of the surfaces $x + y^2 + z^3 = 1$ and $y + z^2 = 1$.

(a) Find a parameterization of C.

take $z = t$

$y + z^2 = 1 \Rightarrow y = 1 - z^2 = 1 - t^2$

$x + y^2 + z^3 = 1 \Rightarrow x = 1 - y^2 - z^3 = 1 - (1-t^2)^2 - t^3$

$= 1 - 1 + 2t^2 - t^4 - t^3$

$= 2t^2 - t^3 - t^4$

$$\boxed{\vec{r}(t) = \langle 2t^2 - t^3 - t^4, 1 - t^2, t \rangle}$$

(b) Let D be the segment of C starting at $(0,1,0)$ and ending at $(0,0,1)$ (oriented toward $(0,0,1)$). Compute $\int_D (2z^2 - x)\,dz + e^z\,dy$.

$\vec{r}(0) = \langle 0,1,0 \rangle$

$\vec{r}(1) = \langle 0,0,1 \rangle$

$\frac{dz}{dt} = 1$, $\frac{dy}{dt} = -2t$

$$\int_C (2z^2 - x)dz + e^z dy =$$

$$= \int_0^1 \left(2t^2 - (2t^2 - t^3 - t^4)\right) \cdot 1 + e^t(-2t)\, dt$$

$$= \int_0^1 t^3 + t^4 - 2te^t\, dt$$

Extra space for work.

integration by parts

$$= \left[\frac{t^4}{4} - \frac{t^5}{5}\right]_0^1 - 2\int_0^1 te^t\,dt$$

$$= \left[\frac{1}{4} + \frac{1}{5} - 0\right] - 2\left[\left[te^t\right]_0^1 - \int_0^1 e^t\,dt\right]$$

$$= \frac{5+4}{20} - 2\left[(e-0) - \left[e^t\right]_0^1\right]$$

$$= \frac{9}{20} - 2\left[e - (e-1)\right]$$

$$= \frac{9}{20} - 2 = \boxed{-\frac{11}{20}}$$