

University of Ottawa  
Dept. of mathematics and Statistics  
Calculus III for engineers  
MAT 2322 3X - Spring-Summer 2015  
Practice Exam for the Final  
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*They will be approximately 17 questions among which 10 are multiple choice questions. Don't forget to review the midterms and the midterms practice tests.*

I will update the solution if I find a typo.

1. Let  $f(x, y) = x \ln(x - 2y)$ . Find the equation of the tangent plane to the graph of  $f$  at the point  $(3, 1, 0)$ .

**Answer:**  $z = 3x - 6y - 3$ .

2. Find the total arclength of the curve given by

$$\vec{r}(t) = (\cos(t), \sin(t), \ln(\cos(t))) \quad , \quad 0 \leq t \leq \pi/4.$$

**Answer:**  $\ln(1 + \sqrt{2})$ .

3. Let  $f(x, y) = \cos(x^2y - 2y^2)$ . Find the directional derivative of  $f$  at the point  $(0, 0)$ ,  $D_{\vec{u}}f(0, 0)$ , in the direction  $\vec{u} = -\vec{i} + \sqrt{3}\vec{j}$ .

**Answer:** 0.

4. Consider the vector field  $\vec{F}(x, y, z) = (xy^2 + xz^2)\vec{i} + (x^2y + yz^2)\vec{j} + (y^2z + x^2z)\vec{k}$ .

(a) Show that  $\vec{F}$  is conservative.

(b) (Bonus) Find a potential function for  $\vec{F}$ , i.e. find  $f(x, y, z)$  such that  $\vec{F}(x, y, z) = \vec{\nabla} f(x, y, z)$

**Answer:** (a)  $\text{rot}\vec{F} = \vec{0} \Rightarrow \vec{F}$  conservative.

(b)  $f(x, y, z) = (x^2z^2 + x^2y^2 + y^2z^2)/2 + K$ .

5. Find and classify the critical points of the function  $f(x, y) = (y^2 + x^2)e^{y^2 - x^2}$ .

**Answer:**  $(0, 0)$  local min,  $(\pm 1, 0)$  saddle pts.

6. Compute the following double integral  $\int_0^1 \int_x^1 \cos(\pi y^2) dy dx$ . Hint: inverse the order of integration.

**Answer:** 0.

7. Let  $\vec{F}(x, y) = (e^{x^2} - \cos(y))\vec{i} + (y^3 + x \sin(y) + 2xy)\vec{j}$  and let  $C$  be the closed curve composed of the half-circle  $x^2 + y^2 = 1$ ,  $y \geq 0$ , and the straight segment from the point  $(-1, 0)$  to the point  $(1, 0)$  oriented counterclockwise. Compute the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

**Answer:** Use Green's theorem and polar coords to find  $4/3$ .

8. Let  $E$  be the tri-dimensional solid **in the first octant** bounded from outside by the sphere  $x^2 + y^2 + z^2 = 1$  and from inside by the cone  $z = \sqrt{x^2 + y^2}$ . If the mass density of this solid is  $\delta(x, y, z) = x + y$ , find the total mass of this solid?

Hint: use spherical coords.

**Answer:**  $\frac{\pi-2}{16}$ . (Check this...)

9. Let  $S$  be the positively (outward) oriented surface composed of the part of the cone  $z = \sqrt{x^2 + y^2}$  for  $1 \leq z \leq 2$  and consider the vector field

$$\vec{F} = 2xz\vec{i} - yz\vec{j} + (z - 3y^2)\vec{k}.$$

Compute  $\iint_S \vec{F} d\vec{S}$ .

**Answer:**  $31\pi/3$ .

10. Consider the parametric surface  $S$  given by

$$\vec{r}(\theta, \phi) = \cos(\theta) \sin(\phi) \vec{i} + \sin(\theta) \sin(\phi) \vec{j} + \cos(\phi) \vec{k}, \quad 0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

Let  $f(x, y, z) = x^2 + y^2$ . What is the value of the surface integral  $\iint_S f dS$ ?

**Answer:**  $\pi/6$ .

11. Let  $E$  be the solid in the first octant bounded by  $z = x^2 + y^2$  and  $z = 8 - x^2 - y^2$ . If the mass density  $\delta(x, y, z) = x + y$ , find the total mass of this solid.

**Answer:** Use cylindrical coords to find  $256/15$ .

**12.** Find the absolute extrema of the function  $f(x, y) = 2x^2 + y^2$  on the disk centred at the point  $(0,1)$  and radius 2.

**Answer:** Abs Min:  $(0, 0)$ . Abs Maxs:  $(0,-1)$ ,  $(0,3)$ ,  $(-\sqrt{3}, 2)$ ,  $(\sqrt{3}, 2)$ .

13. Consider the vector field  $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ .

(a) Compute the divergence of  $\vec{F}$ , i.e. compute  $\operatorname{div}\vec{F} = \vec{\nabla} \cdot \vec{F}$ .

(b) **Use the divergence thm** to compute the surface (flux) integral  $\int \int_S \vec{F} \cdot d\vec{S}$ , where  $S$  is sphere centred at the origin and radius  $a$  oriented positively. (c) Compute directly  $\int \int_S \vec{F} \cdot d\vec{S}$  for  $S$  and  $\vec{F}$  as above.

**Answer:** (a)  $\operatorname{div}\vec{F} = 3$ .  
(b),(c)  $4\pi a^3$ .

14. Sketch the region of integration and then compute the following double integral

$$\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y) dx dy$$

**Hint:** Use another coords system (polar).

**Answer:**  $2\sqrt{2}/3$ .

15. Consider the vector field  $\vec{F} = 2xe^{x^2} \sin(y)\vec{i} + e^{x^2} \cos(y)\vec{j}$ .

(a) Show that  $\vec{F}$  is conservative.

(b) Find a potential function  $f$ ; i.e. find  $f$  such that  $\nabla f = \vec{F}$ .

(c) Evaluate the line integral  $\int_C \vec{F} \cdot d\vec{r}$ , where  $C$  is the arc of the circle parametrized by  $\vec{r} = \cos(\theta)\vec{i} + \sin(\theta)\vec{j}$ ,  $-\pi/2 \leq \theta \leq \pi/2$ .

**Answer:** (a)  $P_y = Q_x$ .

(b)  $f(x, y) = e^{x^2} \sin(y)$ .

(c) Use the F.T.L.I. to find  $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(\pi/2)) - f(\vec{r}(-\pi/2)) = \text{find the numerical value}$ .

16. Consider the vector field

$$\vec{F}(x, y, z) = (yz^3 - 2y)\vec{i} + (xz^3 + 2x)\vec{j} + (3xyz^2 + z^4)\vec{k}$$

and let  $C$  the circle  $x^2 + y^2 = 9$  on the plane  $z = 0$  oriented positively. Evaluate the line integral  $\int_C \vec{F} \cdot d\vec{r}$ . You can use either a direct computation, or Stokes' theorem. Note that one of these two methods will yield a much simpler computation than the other one, so choose the method carefully.

**Answer:** Stoke's thm in this case gives a simpler computation.  $36\pi$