MATH 135 F 2012: Assignment 5

Due: 8:30 AM, Wed., 2012 Oct. 24 in the dropboxes outside MC 4066
or in the electronic dropbox for students in the online section

Write your answers in the space provided. If you wish to typeset your solutions, use one of the solution templates posted on the course web site.

Family Name:

First Name:

I.D. Number:

Section:

Mark: (For the marker only.)

If you used any references beyond the course text and lectures (such as other texts, discussions with colleagues or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

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1. (a) Find all integer solutions to the linear Diophantine equation $36x + 48y = 18$.

(b) Find all integer solutions to the linear Diophantine equation $36x + 438y = 18$.

(c) Find all positive integer solutions to the linear Diophantine equation $36x + 438y = 18$.

2. (From a collection attributed to Alcuin of York in 775 who is known to have sent a collection of similar problems to Charlemagne.)

   One hundred bushels of grain are distributed among one hundred people in such a way that every man receives three bushels, every woman receives two bushels and every child receives half a bushel. How many men, women and children are there?

3. The following proposition is proved in Section 22.3 of your notes.

   Proposition 1. *If k and ℓ are coprime integers, then the Diophantine equation $kx - \ell y = c$ has infinitely many positive integer solutions.*

   Read the proof and apply it to the case $11x - 7y = 3$. Specifically, what values should the x_0, y_0, n, x and y of the proof take on? You do not need to show your work.

4. Prove or disprove each of the following statements.

   (a) Let $a, b, c \in \mathbb{Z}$. Then $\gcd(a, b) = \gcd(a, c) \cdot \gcd(b, c)$.

   (b) Let a, b, c be nonzero integers. If $\gcd(a, b) \mid c$, then $ax^2 + by^2 = c$ has a solution.

   (c) If S is the complete solution to the Diophantine equation $ax + by = d$, then for any integer c, $cS = \{(cx, cy) \mid (x, y) \in S\}$ is the complete solution to $ax + by = cd$.

5. Prove the following proposition.

Proposition 2. *Let* $a, b \in \mathbb{Z}$*. For every integer* $n \in \mathbb{N}$*, if* $\gcd(a, b) = 1$*, then* $\gcd(a, b^n) = 1$*.*

6. Consider the proposition and proposed proof below.

Proposition 3. *Let a, b and c be integers. If $ax+by=c$ has an even solution, then it has an infinite number of even solutions.*

Proof. (a) Suppose (x_0, y_0) is one particular been solution to $ax+by=c$.

(b) By LDET 2, a complete solution is $x = x_0 + \frac{b}{d}n$, $y = y_0 - \frac{a}{d}n$ where $d = \gcd(a, b)$.

(c) Since $\frac{b}{d} \in \mathbb{Z}$ and x_0 is even, for every choice of even n, $x = x_0 + \frac{b}{d}n$ is even.

(d) Since y_0 is even, choosing $n = 0$ for $y = y_0 - \frac{a}{d}n$ gives y_0 which is even,

(e) Hence,

$$\{(x_0 + \frac{b}{d}n, y_0) \mid n \text{ is an even integer}\}$$

is an infinite set of solutions to $ax+by=c$.

□

If the proof is correct, write "CORRECT" in the space provided. If the proof is incorrect, identify the first sentence in the proof that contains an error and explain why this is an error.

7. Given any rational number r, prove that there exist coprime integers p and q, with $q \neq 0$, so that $r = \frac{p}{q}$.