MATH 135 F 2012: Assignment 3

Due: 8:30 AM, Wed., 2012 Oct. 3 in the dropboxes outside MC 4066
or in the electronic dropbox for students in the online section

Write your answers in the space provided. If you wish to typeset your solutions, use one of the solution templates posted on the course web site.

Family Name:

First Name:

I.D. Number:

Section:

Mark: (For the marker only.)

If you used any references beyond the course text and lectures (such as other texts, discussions with colleagues or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Use induction to prove the following statements. In each case, explicitly identify the statement $P(n)$ that you are working with.

1. For all $n \in \mathbb{Z}$, $n \geq 0$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

2. For all $n \in \mathbb{N}$, $4 \mid (5^n - 1)$.

3. Consider the product

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)$$

(a) What is the value of this product for $n = 2, 3, 4, 5$.

(b) Conjecture a value for the product as a function of n.

(c) Use induction to prove your conjecture.

4. The **Fibonacci Numbers** are the sequence of positive integers $f_0, f_1, f_2, f_3, \ldots$ defined inductively by

   - $f_0 = 1$ and $f_1 = 1$ and
   - for all integers $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$.

   (a) Complete this table, calculating f_n for all $0 \leq n \leq 8$.

n	0	1	2	3	4	5	6	7	8
f_n	1	1							

   (b) For all integer $n \geq 0$, let $S_n = \sum_{i=0}^{n} f_i$. Complete this table, calculating S_n for all $0 \leq n \leq 8$.

n	0	1	2	3	4	5	6	7	8
S_n	1	2							

   (c) Prove that for all integers $n \geq 0$,

$$\sum_{i=0}^{n} f_i = f_{n+2} - 1.$$

5. A sequence $\{x_n\}$ is defined by $x_1 = 11$, $x_2 = 23$ and $x_n = x_{n-1} + 12x_{n-2}$ for all $n \geq 3$. For all $n \in \mathbb{N}$, $x_n = 2 \cdot 4^n - (-3)^n$.

6. In Chapter 14, Proposition 1 was proved by contradiction. Prove the same proposition, this time using induction.

   Proposition 1. *If $n \geq 2$ is an integer, then n can be written as a product of primes.*

7. What is wrong with the following "proof" that all horses are the same colour?

   Let $P(n)$ be the proposition that all the horses in a set of n horses are the same colour. Clearly, $P(1)$ is true. Now assume that $P(k)$ is true, so that all the horses in any set of k horses are the same colour. Consider any set of $k+1$ horses; number these as horses $1, 2, 3, \ldots, k, k+1$. Now the first k of these horses all must have the same colour, and the last k of these must also have the same colour. Since the set of the first k horses and the set of the last k horses overlap, all $k+1$ must be the same colour. This shows that $P(k+1)$ is true and finishes the proof by induction.