Solution to Midterm Examination 1 (version B)

MAT 1322-3X, Summer 2015

1. (4 marks) Let R be the region above the graph of $y = \frac{1}{2}x^2$ and under the graph of $y = 3x - 4$. S is the solid obtained by revolving R about the ***y*-axis**. Find the volume of this solid.

Solution. Find the inverse function: $x = \sqrt{2y}$ and $x = \frac{1}{3}(y + 4)$. Let $\sqrt{2y} = \frac{1}{3}(y + 4)$. $18y = y^2 + 8y + 16$. Then $y^2 - 10y + 16 = 0$, $y = 2, 8$. The intersection points of these two curves are (2, 2) and (4, 8). The volume of S is

$$V = \pi\int_2^8\left(\left(\sqrt{2y}\right)^2 - \left(\frac{1}{3}(y+4)\right)^2\right)dy = \pi\left[y^2 - \frac{1}{9}\left(\frac{y^3}{3} + 4y^2 + 16y\right)\right]_{y=2}^{8} = 4\pi.$$

This can also be done by the method of cylindrical shells:

$$V = 2\pi\int_2^4 x\left(3x - 4 - \frac{x^2}{2}\right)dx = 4\pi.$$

2. (4 marks) A bucket with weight 50 kg is lifted from the ground to the top of a building 20 meters high with a rope that weighs 2 kg / m at a constant speed. Because the bucket is leaking, when the bucket is at a height h meters from the ground, the weight of the bucket is $(50 – h)$ kg. Find the work (in Joules) needed to lift the bucket to the top of the building. (Let $g = 9.81$ m/sec^2 be the acceleration of the gravity).

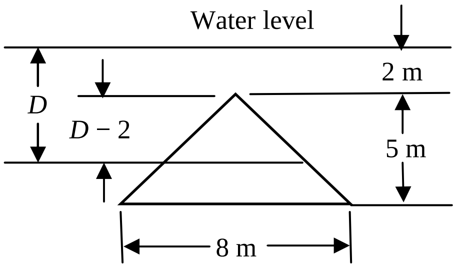
Solution. At height h, the weight of the bucket and the loop is $((50 – h) + 2(20 – h))g = (90 - 3h)$ N. When the bucket is lifted from height h to $h + dh$, the work needed is $dW = (90 – 3h)\, g dh$. The total weight needed to lift the bucket to the top of the building is $W = g\int_0^{20}(90 - 3h)dh \approx 11772$ J.

3. (3 marks) Recall that the length of the arc $y = f(x)$, $a \le x \le b$ is $L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$. Find the length of the arc $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \le x \le 2$.

Solution. $y' = x^2 - \frac{1}{4x^2}$, $(y')^2 = x^4 - \frac{1}{2} + \frac{1}{16x^4}$, $1 + (y')^2 = x^4 + \frac{1}{2} + \frac{1}{16x^4} = \left(x^2 + \frac{1}{4x^2}\right)^2$.

Hence, the length of the arc is $L = \int_1^2\left(x^2 + \frac{1}{4x^2}\right)dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_{x=1}^{2} = \frac{7}{3} + \frac{1}{8} = \frac{59}{24}$.

4. (4 marks) Find the force, in Newtons, acting on a triangular surface submerged into water as shown in the following figure:



(The density of water is $\delta = 1000$ kg/m^3, and g = 9.81).

Solution. Consider a slice with depth D of thickness dD. The width w of the slice is found by similar triangles: $\frac{w}{8} = \frac{D-2}{5}$, $D = \frac{8}{5}(D-2)$. The area is $dD = \frac{8}{5}(D-2)dD$. The force acting on this slice is $dF = \frac{8}{5}(D-2)\delta g D dD$. The total force on this surface is

$$F = \frac{8000g}{5}\int_2^7 (D-2)D\,dD \approx 1.05 \times 10^6 \text{ Newtons.}$$

You may also choose a layer x meters above the bottom of the triangle. Then the area is found by $dA = \frac{8}{5}(5-x)dx$. Then $dF = \frac{8}{5}(5-x)(7-x)dx$, and the total force is

$$F = \frac{4000g}{3}\int_0^5 (5-x)(7-x)\,dx \approx 1.05 \times 10^6 \text{ Newtons.}$$

5. (5 marks) Consider improper integral $\int_0^1 \frac{1}{\sqrt{x}(x+1)}dx$.

(a) (2 marks) Use the comparison test to show that this improper integral is convergent.

(b) (3 marks) Use the definition of improper integrals to find the value of this improper integral.

Solution. (a) When $x > 0$, $(x + 1)\sqrt{x} > \sqrt{x}$. Then $\frac{1}{\sqrt{x}(x+1)} < \frac{1}{\sqrt{x}}$.

Since $\int_0^1 \frac{1}{\sqrt{x}}dx = \int_0^1 \frac{1}{x^{1/2}}dx$ converges. Improper integral $\int_0^1 \frac{1}{\sqrt{x}(x+1)}dx$ converges.

(b) By definition, $\int_0^1 \frac{1}{\sqrt{x}(x+1)}dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}(x+1)}dx$.

Let $u = \sqrt{x}$. Then $u' = \frac{1}{2\sqrt{x}}$.

$$\int_a^1 \frac{1}{\sqrt{x}(x+1)}dx = \int_{\sqrt{a}}^1 \frac{1}{\sqrt{x}(x+1)}\left(2\sqrt{x}\right)du = \int_{\sqrt{a}}^1 \frac{2}{u^2+1}du = 2\left[\arctan u\right]_{u=\sqrt{a}}^1 = 2(\arctan 1 - \arctan\sqrt{a}).$$

When a approaches 0^+,

$$\int_0^1 \frac{1}{\sqrt{x}(x+1)}dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}(x+1)}dx = 2\arctan 1 - \lim_{a\to 0^+}\arctan\sqrt{a} = 2\arctan 1 = \frac{\pi}{2}.$$

6. (4 marks) Let R be the region between the graph of $y = \frac{1}{x}$ and the x-axis, $1 \le x \le 2$. Assuming it has a uniform density ρ. Find the coordinates of the center of mass of this region.

Solution. Without loss of generality, we may assume $\rho = 1$. The mass $m = \int_1^2 \frac{1}{x}dx = \ln 2$.

The moments: $M_x = \frac{1}{2}\int_1^2 \frac{1}{x^2}dx = \frac{1}{4}$, $M_y = \int_1^2 x\frac{1}{x}dx = 1$. Hence, $\overline{x} = \frac{M_y}{m} = \frac{1}{\ln 2}$, $\overline{y} = \frac{M_x}{m} = \frac{1}{4\ln 2}$. The center of mass is $\left(\frac{1}{\ln 2}, \frac{1}{4\ln 2}\right)$.