Solution to Midterm Examination 1 (version A)

MAT 1322-3X, Summer 2015
Total = 20 marks

1. (4 marks) Let R be the region above the graph of $y = x^2$ and under the graph of $y = 3x - 2$. S is the solid obtained by revolving R about the ***y*-axis**. Find the volume of this solid.

Solution. Find the inverse function: $x = \sqrt{y}$ and $x = \frac{1}{3}(y+2)$. Let $\sqrt{y} = \frac{1}{3}(y+2)$.
$9y = y^2 + 4y + 4$. Then $y^2 - 5y + 4 = 0$, $y = 1, 4$. The intersection points of these two curves are (1, 1) and (2, 4). The volume of S is

$$V = \pi\int_1^4\left(\left(\sqrt{y}\right)^2 - \left(\frac{1}{3}(y+2)\right)^2\right)dy = \pi\left[\frac{y^2}{2} - \frac{1}{9}\left(\frac{y^3}{3} + 2y^2 + 4y\right)\right]_{y=1}^4 = \frac{1}{2}\pi.$$

This can also be done by the method of cylindrical shells:

$$V = 2\pi\int_1^2 x(3x - 2 - x^2)dx = 2\pi\left[x^3 - x^2 - \frac{x^4}{4}\right]_{x=1}^2 = \frac{\pi}{2}.$$

2. (4 marks) A bucket with weight 20 kg is lifted from the ground to the top of a building 10 meters high with a rope that weighs 0.5 kg / m at a constant speed. Because the bucket is leaking, when the bucket is at a height h meters from the ground, the weight of the bucket is $(20 - 0.5h)$ kg. Find the work (in Joules) needed to lift the bucket to the top of the building. (Let $g = 9.81$ m/sec^2 be the acceleration of the gravity).

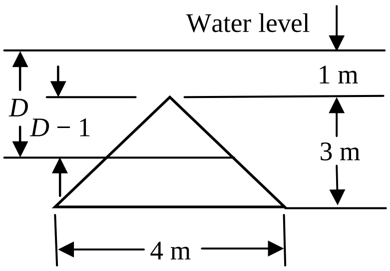
Solution. At height h, the weight of the bucket and the loop is $((20 - 0.5h) + 0.5(10 - h))g = (25 - h)$ N. When the bucket is lifted from height h to $h + dh$, the work needed is $dW = (25 - h)\, gdh$. The total weight needed to lift the bucket to the top of the building is
$W = g\int_0^{10}(25 - h)dh \approx 1962$ J.

3. (3 marks) Recall that the length of the arc $y = f(x)$, $a \le x \le b$ is $L = \int_a^b\sqrt{1 + (f'(x))^2}\,dx$. Find the length of the arc $y = \frac{x^3}{12} + \frac{1}{x}$, $1 \le x \le 3$.

Solution. $y' = \frac{x^2}{4} - \frac{1}{x^2}$, $(y')^2 = \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$, $1 + (y')^2 = \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$.

Hence, the length of the arc is $L = \int_1^3\left(\frac{x^2}{4} + \frac{1}{x^2}\right)dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_{x=1}^3 = \frac{17}{6}$.

4. (4 marks) Find the force, in Newtons, acting on a triangular surface submerged into water as shown in the following figure:



(The density of water is δ = 1000 kg/m^3, and g = 9.81).

Solution. Consider a slice with depth D of thickness dD. The width w of the slice is found by similar triangles: $\frac{w}{4} = \frac{D-1}{3}$, $D = \frac{4}{3}(D-1)$. The area is $dA = \frac{4}{3}(D-1)dD$. The force acting on this slice is $dF = \frac{4}{3}(D-1)\delta g D dD$. The total force on this surface is

$$F = \frac{4000g}{3}\int_1^4 (D-1)D dD \approx 1.77 \times 10^5 \text{ Newtons.}$$

You may also choose a layer x meters above the bottom of the triangle. Then the area is found by $dA = \frac{4}{3}(3-x)dx$. Then $dF = \frac{4}{3}(3-x)(4-x)dx$, and the total force is

$$F = \frac{4000g}{3}\int_0^3 (3-x)(4-x)dx \approx 1.77 \times 10^5 \text{ Newtons.}$$

5. (5 marks) Consider improper integral $\int_0^1 \frac{1}{x+\sqrt{x}}dx$.

(a) (2 marks) Use the comparison test to show that this improper integral is convergent.

(b) (3 marks) Use the definition of improper integrals to find the value of this improper integral .

Solution. (a) When $x > 0$, $x + \sqrt{x} > \sqrt{x}$. Then $\frac{1}{x+\sqrt{x}} < \frac{1}{\sqrt{x}}$.

Since $\int_0^1 \frac{1}{\sqrt{x}}dx = \int_0^1 \frac{1}{x^{1/2}}dx$ converges. Improper integral $\int_0^1 \frac{1}{x+\sqrt{x}}dx$ converges.

(b) By definition, $\int_0^1 \frac{1}{x+\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x+\sqrt{x}}\,dx$.

Let $u = \sqrt{x}$. Then $u' = \frac{1}{2\sqrt{x}}$.

$$\int_a^1 \frac{1}{x+\sqrt{x}}\,dx = \int_{\sqrt{a}}^1 \frac{1}{x+\sqrt{x}}\left(2\sqrt{x}\right)du = \int_{\sqrt{a}}^1 \frac{2}{u+1}\,du = 2\left[\ln(u+1)\right]_{u=\sqrt{a}}^1 = 2(\ln 2 - \ln(\sqrt{a}+1)).$$

When a approaches 0^+, $\int_0^1 \frac{1}{x+\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x+\sqrt{x}}\,dx = 2\ln 2 - \lim_{a\to 0^+}\ln(\sqrt{a}+1) = 2\ln 2.$

6. (4 marks) Let R be the region between the graph of $y = \frac{1}{x}$ and the x-axis, $1 \le x \le 2$. Assuming it has a uniform density ρ. Find the coordinates of the center of mass of this region.

Solution. Without loss of generality, we may assume $\rho = 1$. The mass $m = \int_1^2 \frac{1}{x}\,dx = \ln 2$.

The moments: $M_x = \frac{1}{2}\int_1^2 \frac{1}{x^2}\,dx = \frac{1}{4}$, $M_y = \int_1^2 x\frac{1}{x}\,dx = 1$. Hence, $\bar{x} = \frac{M_y}{m} = \frac{1}{\ln 2}$, $\bar{y} = \frac{M_x}{m} = \frac{1}{4\ln 2}$. The center of mass is $\left(\frac{1}{\ln 2}, \frac{1}{4\ln 2}\right)$.