Chapter 3

Rigid Bodies: Equivalent Systems of Forces

Contents

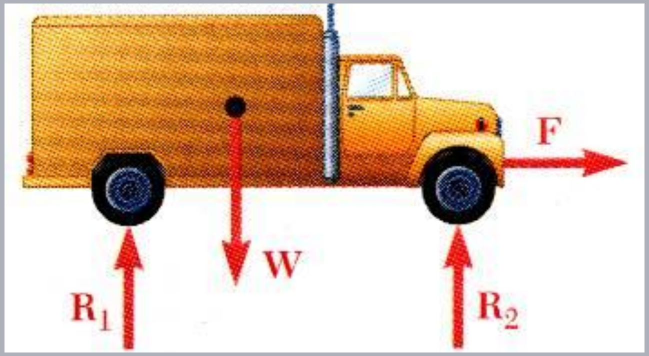
Introduction

- Treatment of a body as a single particle is not always possible. In general, the size of the body and the specific points of application of the forces must be considered.
- Most bodies in elementary mechanics are assumed to be rigid, i.e., the actual deformations are small and do not affect the conditions of equilibrium or motion of the body.
- Current chapter describes the effect of forces exerted on a rigid body and how to replace a given system of forces with a simpler equivalent system.
  - moment of a force about a point
  - moment of a force about an axis
  - moment due to a couple
- Any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple.

External and Internal Forces

- Forces acting on rigid bodies are divided into two groups:
  - External forces
  - Internal forces

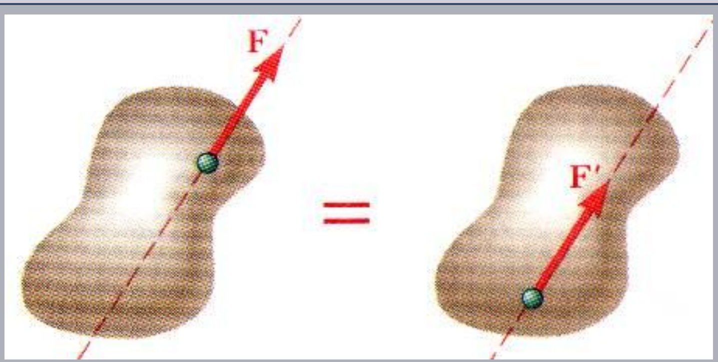
- External forces are shown in a free-body diagram.



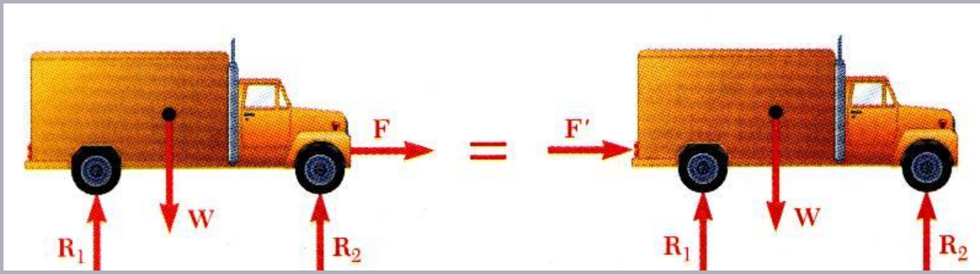
- If unopposed, each external force can impart a motion of translation or rotation, or both.

Principle of Transmissibility: Equivalent Forces

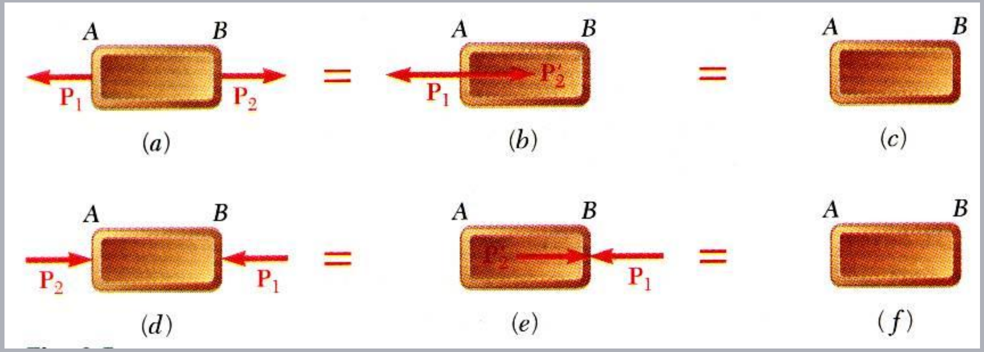
- *Principle of Transmissibility* - Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action.
  NOTE: **F** and **F'** are equivalent forces.



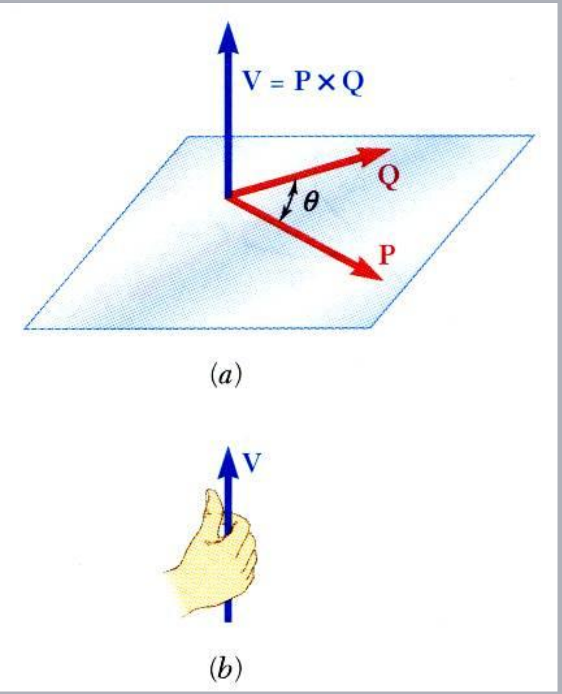
- Moving the point of application of the force **F** to the rear bumper does not affect the motion or the other forces acting on the truck.



- Principle of transmissibility may not always apply in determining internal forces and deformations.



Vector Product of Two Vectors



- Concept of the moment of a force about a point is more easily understood through applications of the *vector product* or *cross product*.

- Vector product of two vectors ***P*** and ***Q*** is defined as the vector ***V*** which satisfies the following conditions:
  1. Line of action of ***V*** is perpendicular to plane containing ***P*** and ***Q***.
  2. Magnitude of ***V*** is $V = PQ\sin\theta$
  3. Direction of ***V*** is obtained from the right-hand rule.

- Vector products:
  - are not commutative, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
  - are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$
  - are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

Moment of a Force About a Point

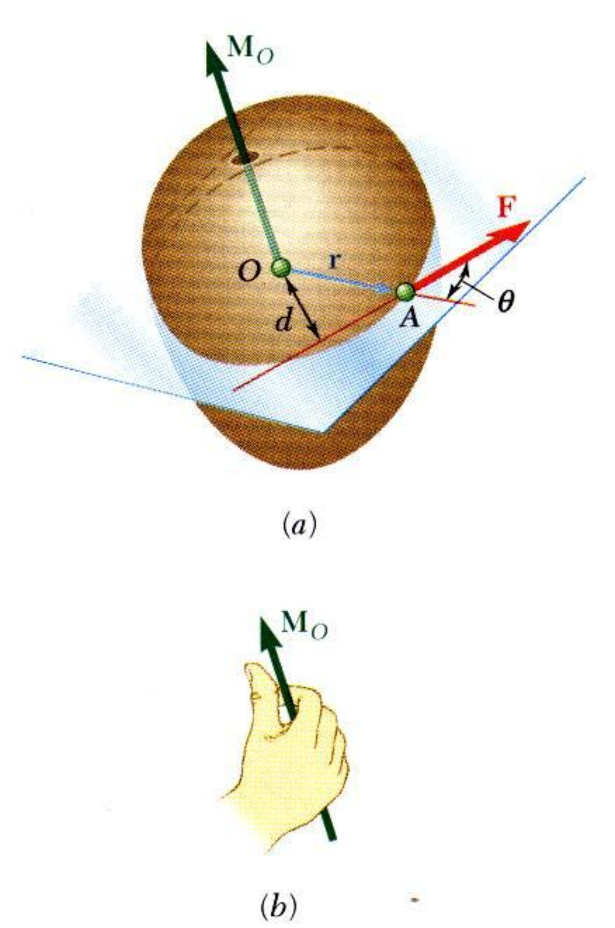
- A force vector is defined by its magnitude and direction. Its effect on the rigid body also depends on it point of application.
- The *moment* of ***F*** about *O* is defined as

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

- The moment vector $\boldsymbol{M_O}$ is perpendicular to the plane containing *O* and the force ***F***.
- Magnitude of $\boldsymbol{M_O}$ measures the tendency of the force to cause rotation of the body about an axis along $\boldsymbol{M_O}$.

$$M_O = rF\sin\theta = Fd$$

  The sense of the moment may be determined by the right-hand rule.
- Any force ***F'*** that has the same magnitude and direction as ***F***, is *equivalent* if it also has the same line of action and therefore, produces the same moment.



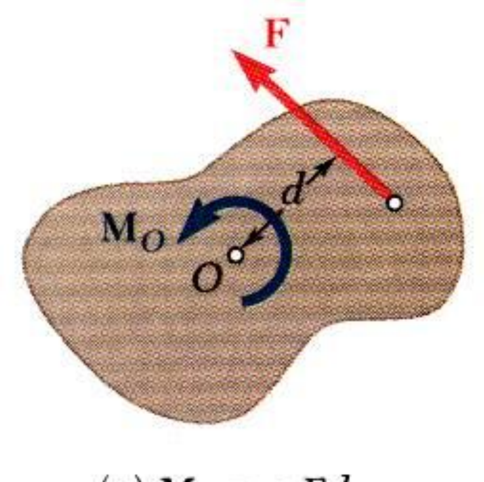
Moment of a Force About a Point

- *Two-dimensional structures* have length and breadth but negligible depth and are subjected to forces contained in the plane of the structure.

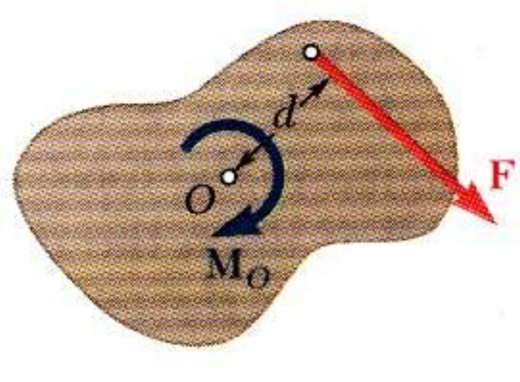
- The plane of the structure contains the point *O* and the force $\boldsymbol{F}$. $\boldsymbol{M_O}$, the moment of the force about *O* is perpendicular to the plane.

- If the force tends to rotate the structure clockwise, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is positive.

- If the force tends to rotate the structure counterclockwise, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is negative.



(a) $\mathbf{M}_O = +Fd$



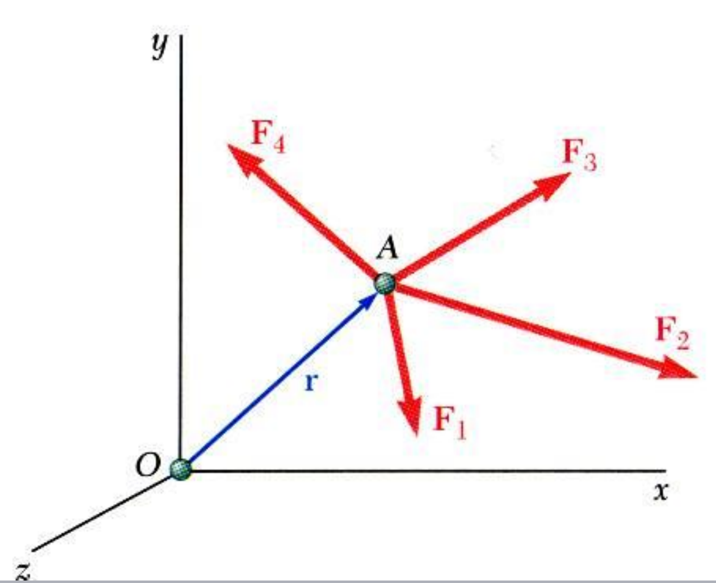
(b) $\mathbf{M}_O = -Fd$

Varignon's Theorem

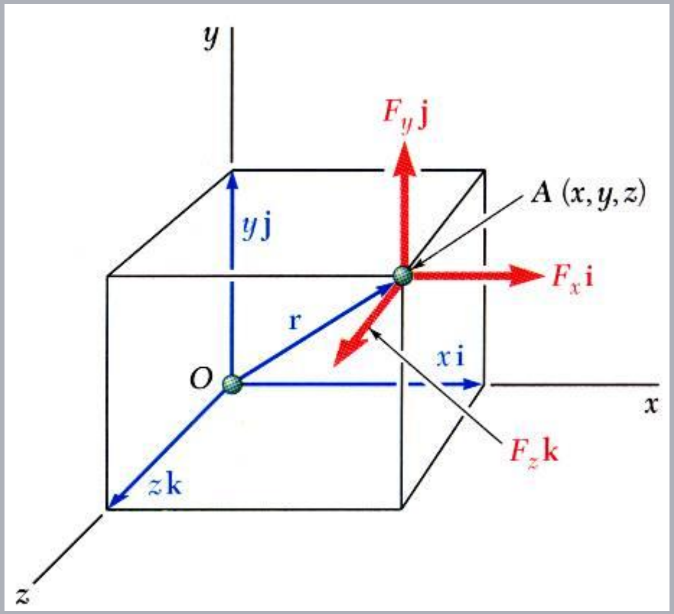
- The moment about a give point *O* of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point *O*.

$$\vec{r} \times \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \cdots$$

- Varigon's Theorem makes it possible to replace the direct determination of the moment of a force ***F*** by the moments of two or more component forces of ***F***.



Rectangular Components of the Moment of a Force



The moment of ***F*** about *O*,

$$\vec{M}_O = \vec{r} \times \vec{F}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(yF_z - zF_y\right)\vec{i} + \left(zF_x - xF_z\right)\vec{j} + \left(xF_y - yF_x\right)\vec{k}$$

Rectangular Components of the Moment of a Force

The moment of ***F*** about *B*,

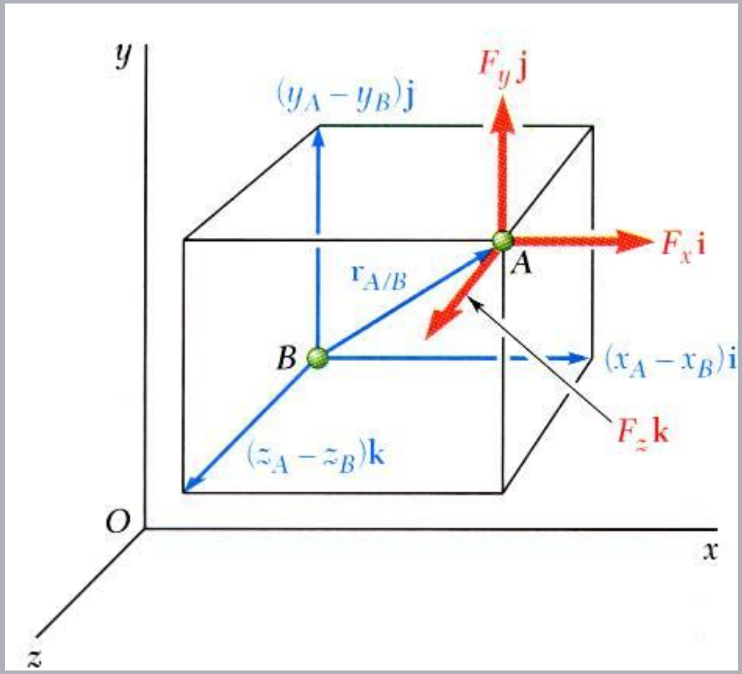
$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$



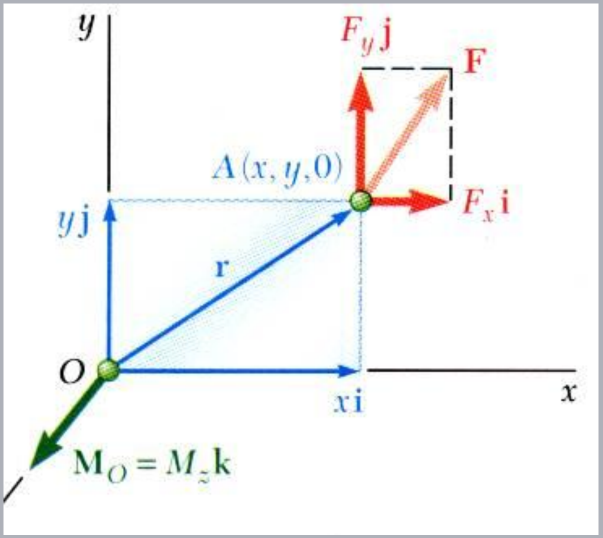
Rectangular Components of the Moment of a Force

For two-dimensional structures,

$$\vec{M}_O = \left(xF_y - yF_z\right)\vec{k}$$

$$M_O = M_Z$$

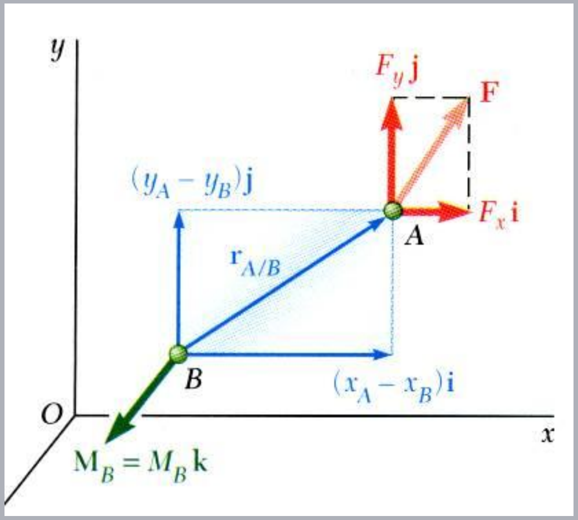
$$= xF_y - yF_z$$



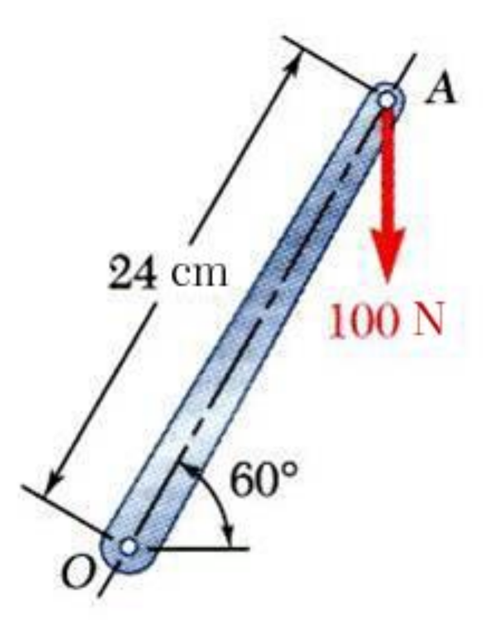
$$\vec{M}_O = \left[(x_A - x_B)F_y - (y_A - y_B)F_z\right]\vec{k}$$

$$M_O = M_Z$$

$$= (x_A - x_B)F_y - (y_A - y_B)F_z$$



Sample Problem 3.1



A 100-N vertical force is applied to the end of a lever which is attached to a shaft at *O*.

Determine:

a) moment about *O*,

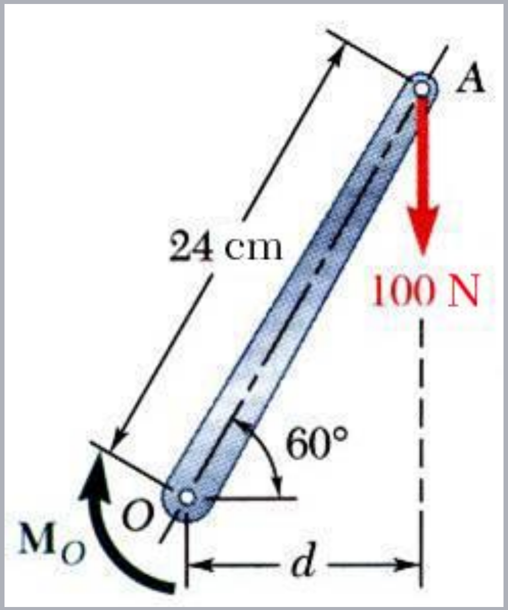
b) horizontal force at *A* which creates the same moment,

c) smallest force at A which produces the same moment,

d) location for a 240-N vertical force to produce the same moment,

e) whether any of the forces from b, c, and d is equivalent to the original force.

Sample Problem 3.1



a) Moment about *O* is equal to the product of the force and the perpendicular distance between the line of action of the force and *O*. Since the force tends to rotate the lever clockwise, the moment vector is into the plane of the paper.

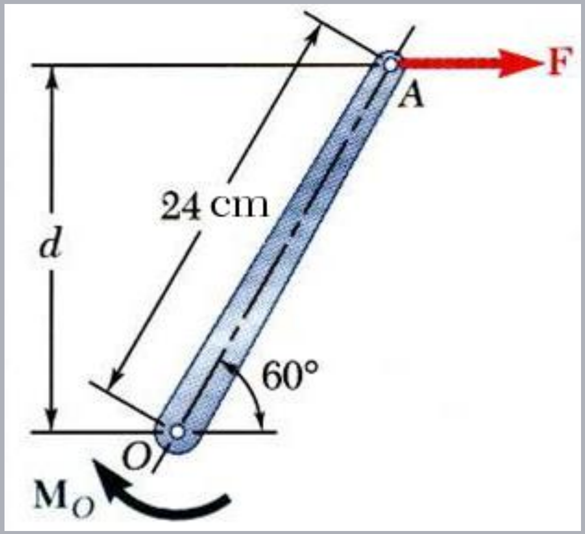
$$M_O = Fd$$

$$d = (24\,\text{cm})\cos 60° = 12\,\text{cm}$$

$$M_O = (100\,\text{N})(12\,\text{cm})$$

$$M_O = 1200\,\text{N}\cdot\text{cm}$$

Sample Problem 3.1



c) Horizontal force at A that produces the same moment,

$$d = (24\text{ cm})\sin 60° = 20.8\text{ cm}$$

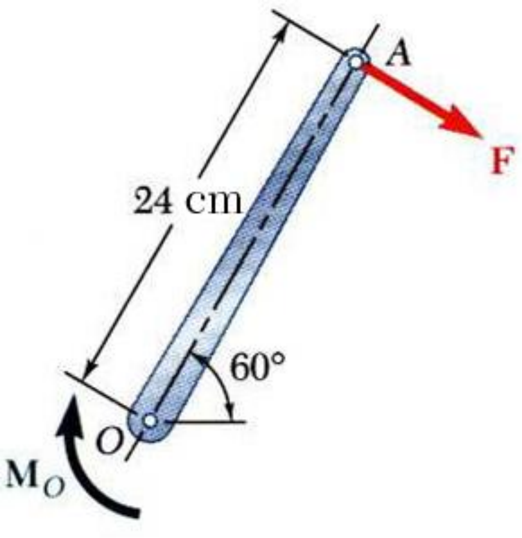
$$M_O = Fd$$

$$1200\text{ N}\cdot\text{cm} = F(20.8\text{ cm})$$

$$F = \frac{1200\text{ N}\cdot\text{cm}}{20.8\text{ cm}}$$

$$\boxed{F = 57.7\text{ N}}$$

Sample Problem 3.1



c) The smallest force *A* to produce the same moment occurs when the perpendicular distance is a maximum or when *F* is perpendicular to *OA*.

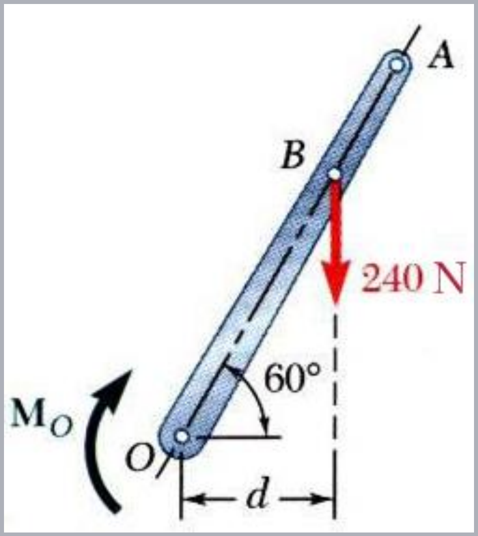
$$M_O = Fd$$

$$1200\ \text{N}\cdot\text{cm} = F(24\ \text{cm})$$

$$F = \frac{1200\ \text{N}\cdot\text{cm}}{24\ \text{cm}}$$

$$\boxed{F = 50\ \text{N}}$$

Sample Problem 3.1



d) To determine the point of application of a 240 lb force to produce the same moment,

$$M_O = Fd$$

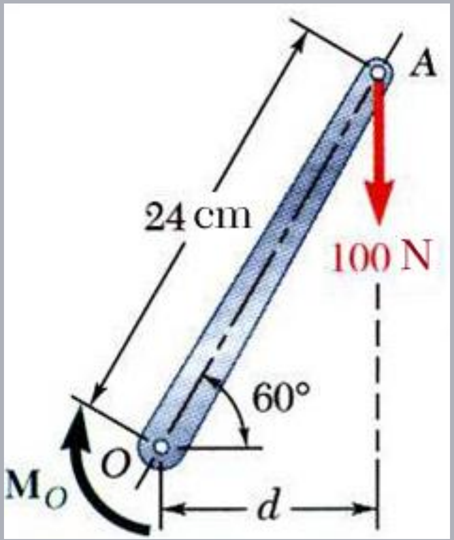
$$1200\ \text{N} \cdot \text{cm} = (240\ \text{N})d$$

$$d = \frac{1200\ \text{N} \cdot \text{cm}}{240\ \text{N}} = 5\ \text{cm}$$

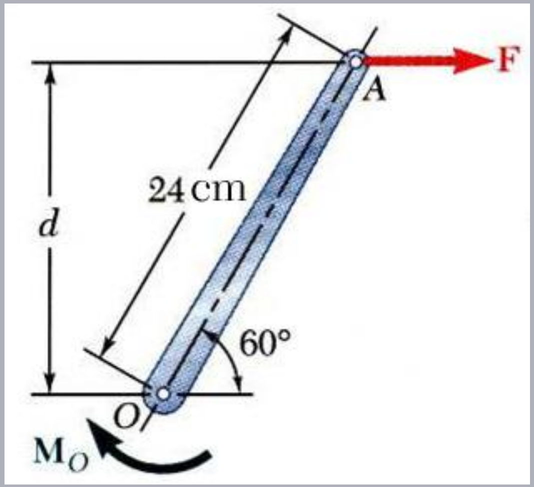
$$OB \cos 60° = 5\ \text{cm}$$

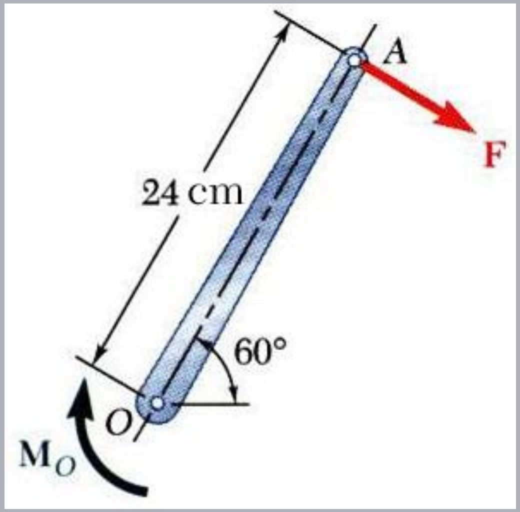
$$\boxed{OB = 10\ \text{cm}}$$

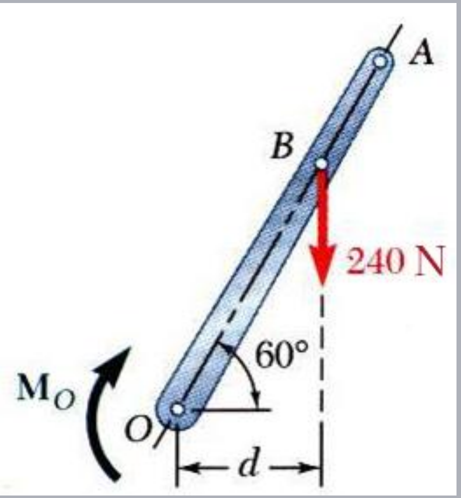
Sample Problem 3.1



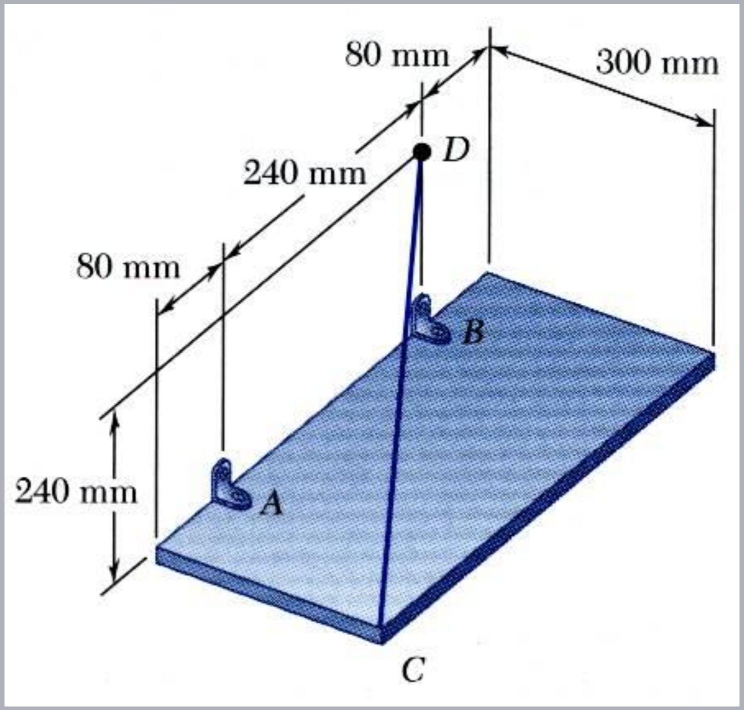
e) Although each of the forces in parts b), c), and d) produces the same moment as the 100 N force, none are of the same magnitude and sense, or on the same line of action. None of the forces is equivalent to the 100 N force.







Sample Problem 3.4



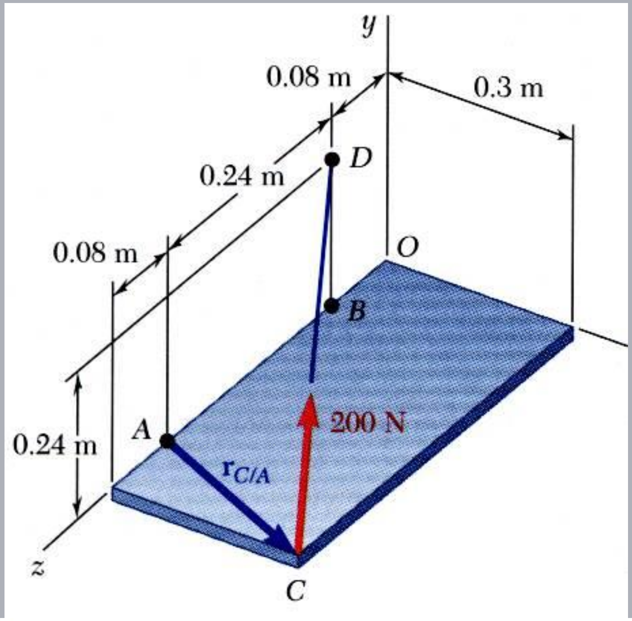
The rectangular plate is supported by the brackets at *A* and *B* and by a wire *CD*. Knowing that the tension in the wire is 200 N, determine the moment about *A* of the force exerted by the wire at *C*.

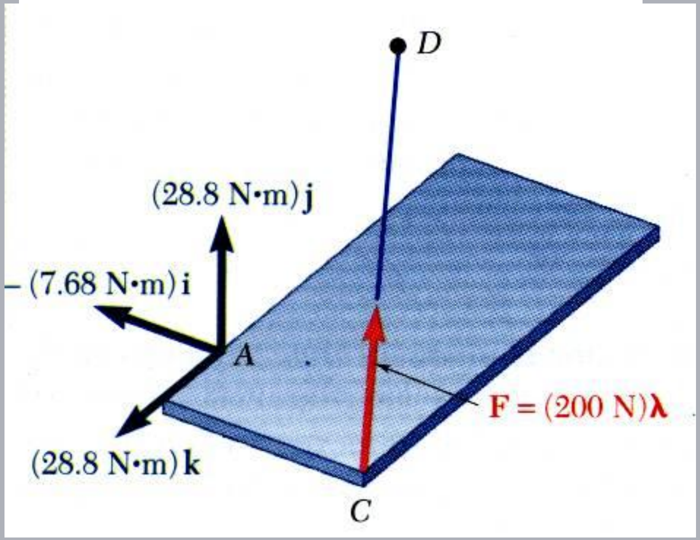
SOLUTION:

The moment M_A of the force *F* exerted by the wire is obtained by evaluating the vector product,

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

Sample Problem 3.4





SOLUTION:

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3\,\text{m})\vec{i} + (0.08\,\text{m})\vec{j}$$

$$\vec{F} = F\vec{\lambda} = (200\,\text{N})\frac{\vec{r}_{C/D}}{r_{C/D}}$$

$$= (200\,\text{N})\frac{-(0.3\,\text{m})\vec{i} + (0.24\,\text{m})\vec{j} - (0.32\,\text{m})\vec{k}}{0.5\,\text{m}}$$

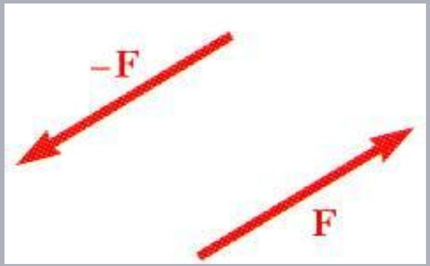
$$= -(120\,\text{N})\vec{i} + (96\,\text{N})\vec{j} - (128\,\text{N})\vec{k}$$

$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\boxed{\vec{M}_A = -(7.68\,\text{N}\cdot\text{m})\vec{i} + (28.8\,\text{N}\cdot\text{m})\vec{j} + (28.8\,\text{N}\cdot\text{m})\vec{k}}$$

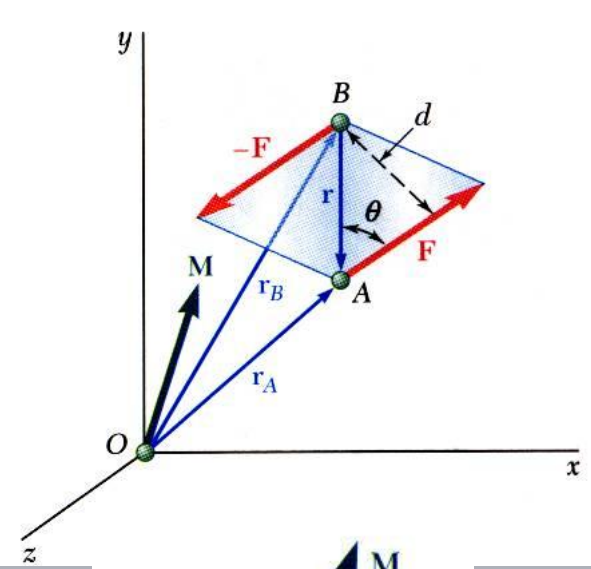
Moment of a Couple

- Two forces ***F*** and -***F*** having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.

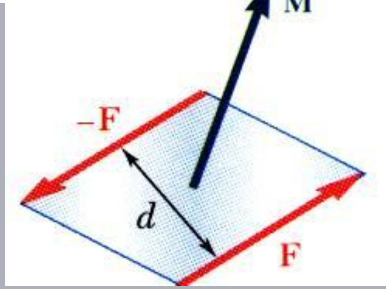


- Moment of the couple,

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right)$$
$$= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$M = rF\sin\theta = Fd$$



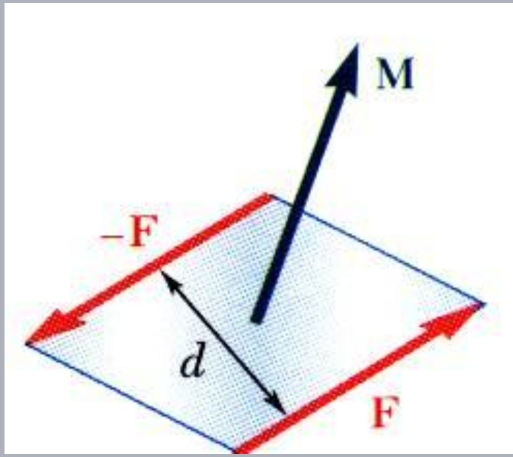
- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.

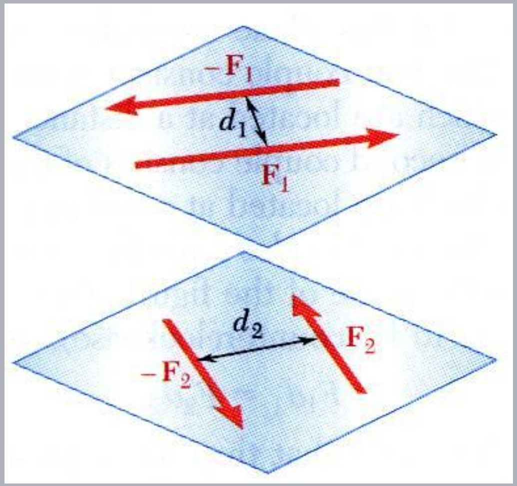


Moment of a Couple

Two couples will have equal moments if

- $F_1 d_1 = F_2 d_2$
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.





Addition of Couples

- Consider two intersecting planes P_1 and P_2 with each containing a couple

$$\vec{M}_1 = \vec{r} \times \vec{F}_1 \text{ in plane } P_1$$

$$\vec{M}_2 = \vec{r} \times \vec{F}_2 \text{ in plane } P_2$$

- Resultants of the vectors also form a couple

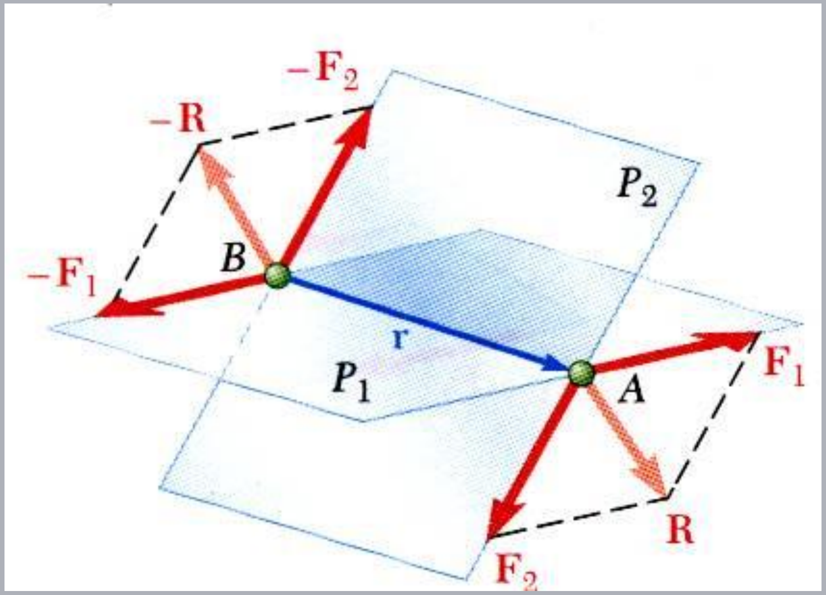
$$\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$

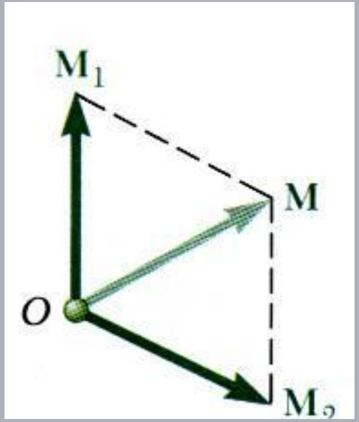
- By Varigon's theorem

$$\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$

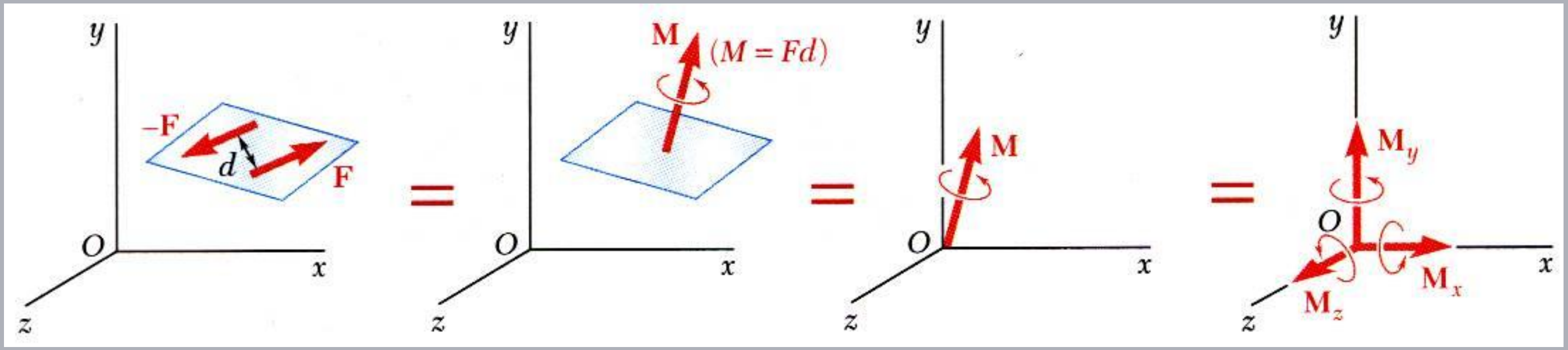
$$= \vec{M}_1 + \vec{M}_2$$

- Sum of two couples is also a couple that is equal to the vector sum of the two couples



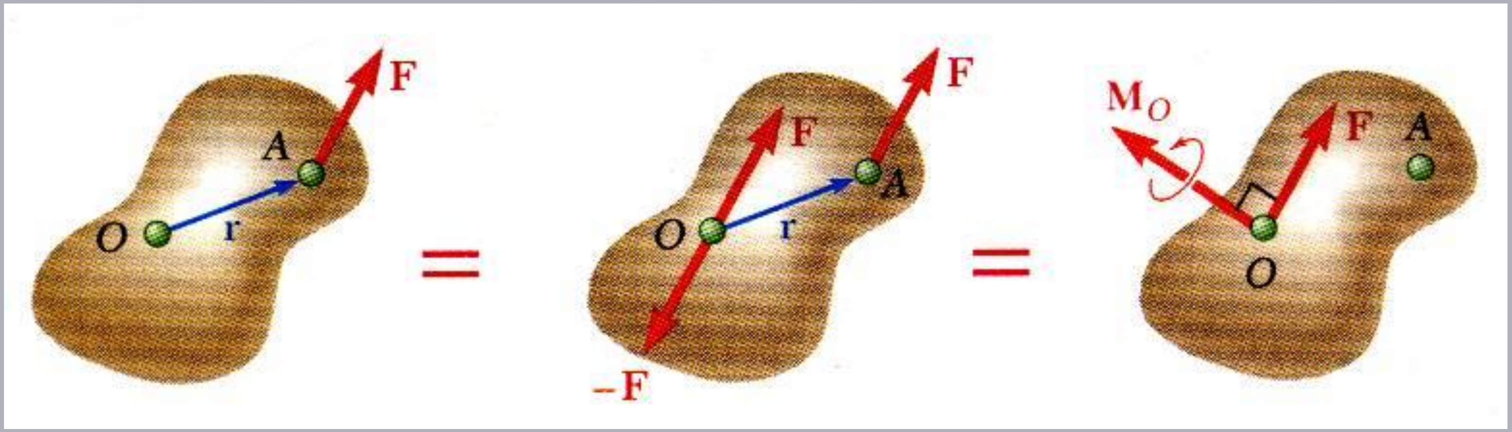


Couples Can Be Represented by Vectors



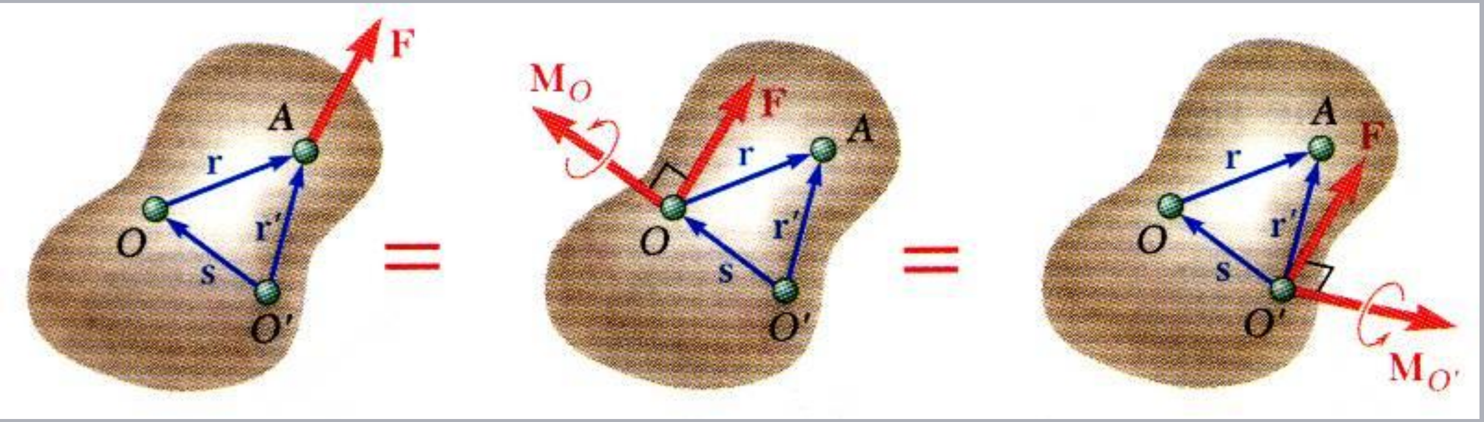
- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.
- *Couple vectors* obey the law of addition of vectors.
- Couple vectors are free vectors, i.e., the point of application is not significant.
- Couple vectors may be resolved into component vectors.

Resolution of a Force Into a Force at *O* and a Couple



- Force vector ***F*** can not be simply moved to *O* without modifying its action on the body.
- Attaching equal and opposite force vectors at *O* produces no net effect on the body.
- The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*.

Resolution of a Force Into a Force at *O* and a Couple



- Moving ***F*** from *A* to a different point *O'* requires the addition of a different couple vector $\boldsymbol{M_{O'}}$

$$\vec{M}_{O'} = \vec{r}\,' \times \vec{F}$$

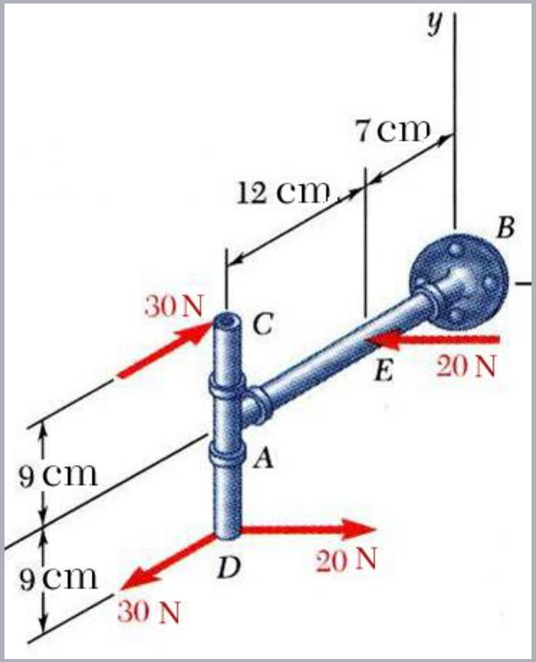
- The moments of ***F*** about O and *O'* are related,

$$\vec{M}_{O'} = \vec{r}\,' \times \vec{F} = (\vec{r} + \vec{s}) \times \vec{F} = \vec{r} \times \vec{F} + \vec{s} \times \vec{F}$$

$$= \vec{M}_O + \vec{s} \times \vec{F}$$

- Moving the force-couple system from *O* to *O'* requires the addition of the moment of the force at *O* about *O'*.

Sample Problem 3.6



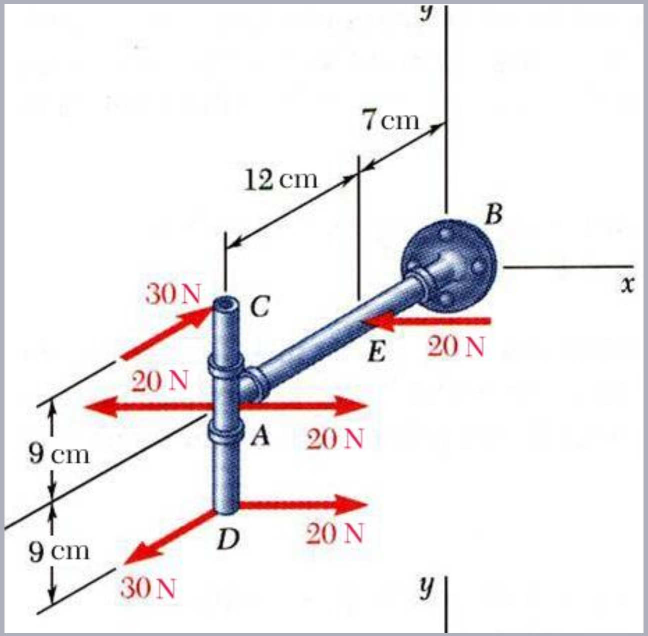
Determine the components of the single couple equivalent to the couples shown.

SOLUTION:

- Attach equal and opposite 20 N forces in the $\pm x$ direction at A, thereby producing 3 couples for which the moment components are easily computed.

- Alternatively, compute the sum of the moments of the four forces about an arbitrary single point. The point D is a good choice as only two of the forces will produce non-zero moment contributions..

Sample Problem 3.6



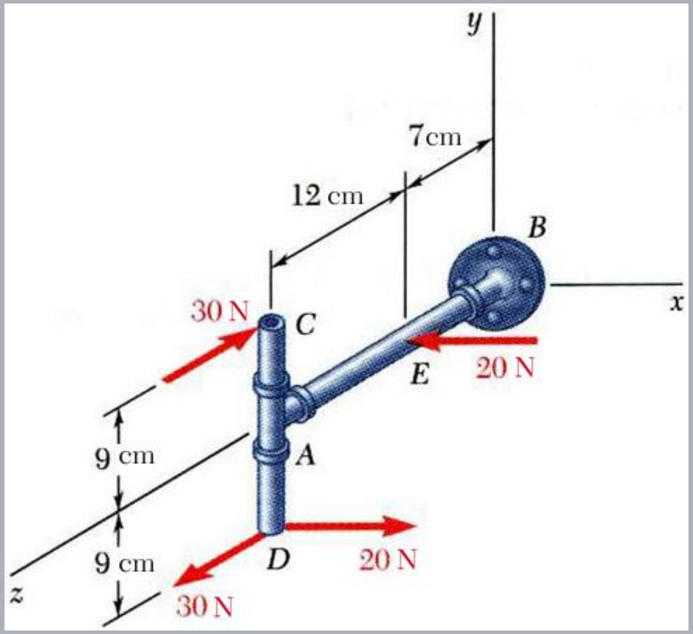
- Attach equal and opposite 20 N forces in the $\pm x$ direction at *A*

- The three couples may be represented by three couple vectors,

$$M_x = -(30\text{ N})(18\text{ cm}) = -540\text{ N}\cdot\text{cm}$$
$$M_y = +(20\text{ N})(12\text{ cm}) = +240\text{N}\cdot\text{cm}$$
$$M_z = +(20\text{ N})(9\text{ cm}) = +180\text{ N}\cdot\text{cm}$$

$$\vec{M} = -(540\text{ N}\cdot\text{cm})\vec{i} + (240\text{N}\cdot\text{cm})\vec{j} + (180\text{ N}\cdot\text{cm})\vec{k}$$

Sample Problem 3.6



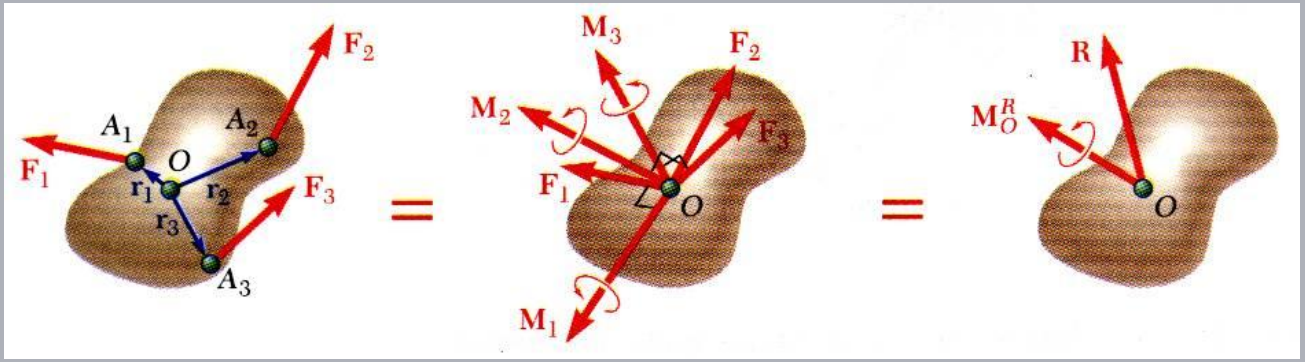
- Alternatively, compute the sum of the moments of the four forces about *D*.

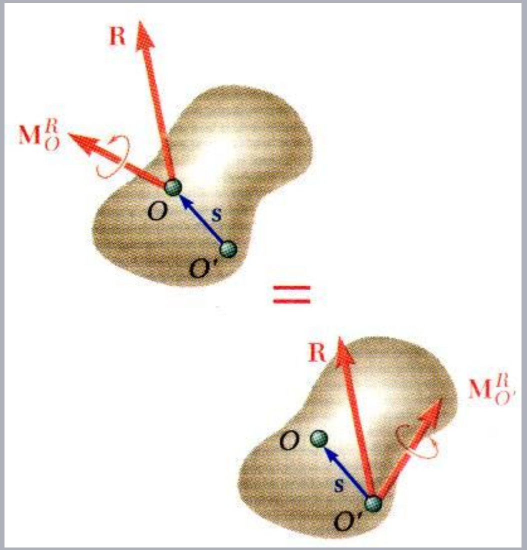
- Only the forces at *C* and *E* contribute to the moment about *D*.

$$\vec{M} = \vec{M}_D = (18\text{ cm})\vec{j} \times (-30\text{ N})\vec{k} + \left[(9\text{ cm})\vec{j} - (12\text{ cm})\vec{k}\right] \times (-20\text{ N})\vec{i}$$

$$\boxed{\vec{M} = -(540\text{ N}\cdot\text{cm})\vec{i} + (240\text{N}\cdot\text{cm})\vec{j} + (180\text{ N}\cdot\text{cm})\vec{k}}$$

System of Forces: Reduction to a Force and Couple





- A system of forces may be replaced by a collection of force-couple systems acting a given point *O*
- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,

$$\vec{R} = \sum \vec{F} \qquad \vec{M}_O^R = \sum \left(\vec{r} \times \vec{F} \right)$$

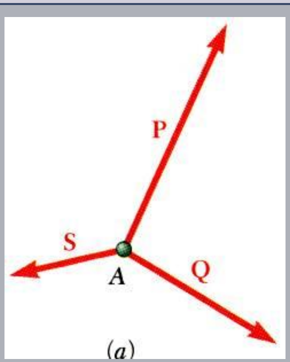
- The force-couple system at *O* may be moved to *O'* with the addition of the moment of ***R*** about *O'* ,

$$\vec{M}_{O'}^R = \vec{M}_O^R + \vec{s} \times \vec{R}$$

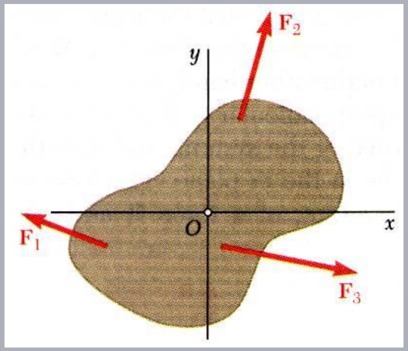
- Two systems of forces are equivalent if they can be reduced to the same force-couple system.

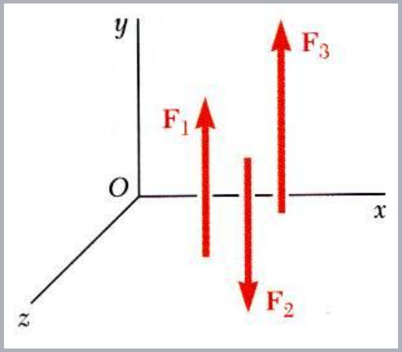
Further Reduction of a System of Forces

- If the resultant force and couple at *O* are mutually perpendicular, they can be replaced by a single force acting along a new line of action.

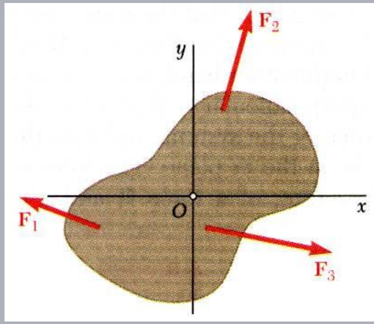


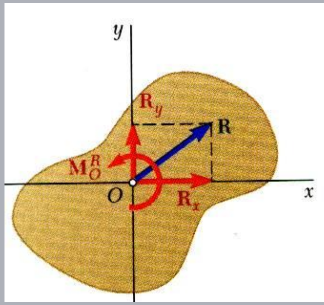
- The resultant force-couple system for a system of forces will be mutually perpendicular if:
  1) the forces are concurrent,
  2) the forces are coplanar, or
  3) the forces are parallel.



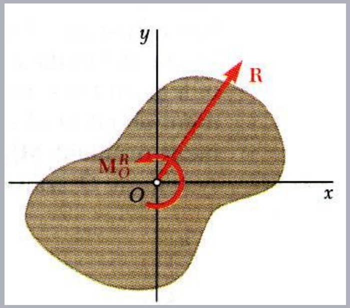


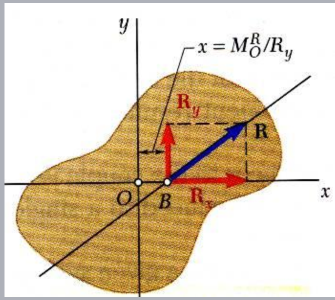
Further Reduction of a System of Forces



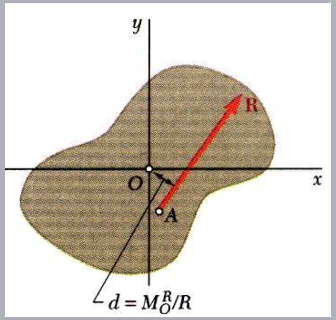


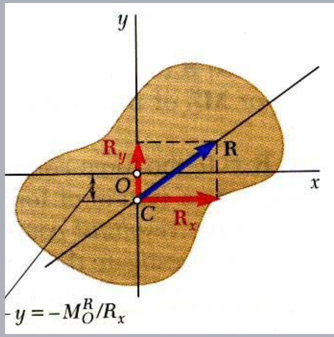
- System of coplanar forces is reduced to a force-couple system $\vec{R}$ and $\vec{M}_O^R$ that is mutually perpendicular.





- System can be reduced to a single force by moving the line of action of $\vec{R}$ until its moment about O becomes $\vec{M}_O^R$

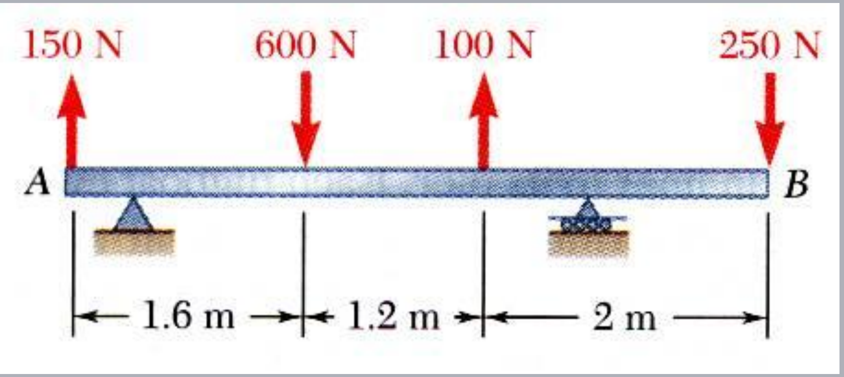




- In terms of rectangular coordinates,

$$xR_y - yR_x = M_O^R$$

Sample Problem 3.8



For the beam, reduce the system of forces shown to (a) an equivalent force-couple system at *A*, (b) an equivalent force couple system at *B*, and (c) a single force or resultant.

Note: Since the support reactions are not included, the given system will not maintain the beam in equilibrium.

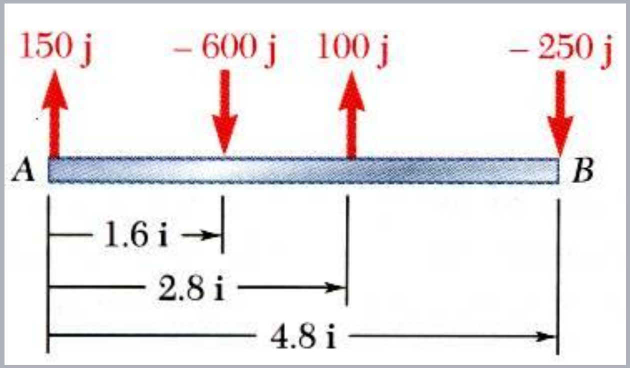
<u>SOLUTION:</u>

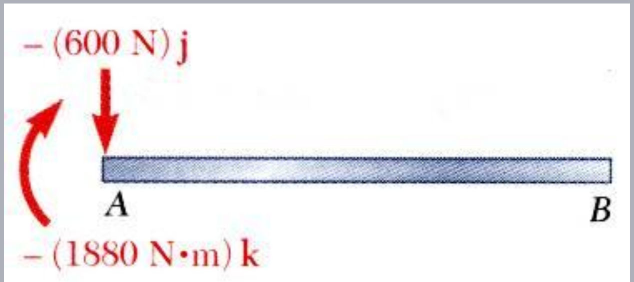
a) Compute the resultant force for the forces shown and the resultant couple for the moments of the forces about *A*.

b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

c) Determine the point of application for the resultant force such that its moment about *A* is equal to the resultant couple at *A*.

Sample Problem 3.8





<u>SOLUTION:</u>

a) Compute the resultant force and the resultant couple at *A*.

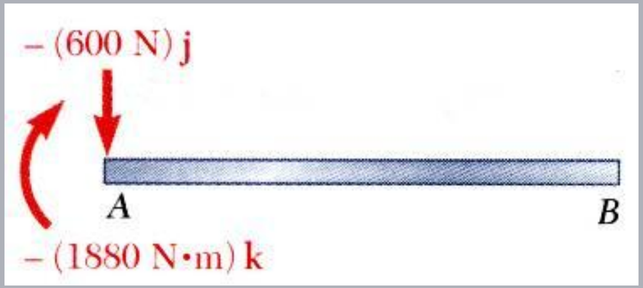
$$\vec{R} = \sum \vec{F}$$
$$= (150\text{ N})\vec{j} - (600\text{ N})\vec{j} + (100\text{ N})\vec{j} - (250\text{ N})\vec{j}$$

$$\boxed{\vec{R} = -(600\text{ N})\vec{j}}$$

$$\vec{M}_A^R = \sum(\vec{r} \times \vec{F})$$
$$= (1.6\vec{i}) \times (-600\vec{j}) + (2.8\vec{i}) \times (100\vec{j}) + (4.8\vec{i}) \times (-250\vec{j})$$

$$\boxed{\vec{M}_A^R = -(1880\text{ N}\cdot\text{m})\vec{k}}$$

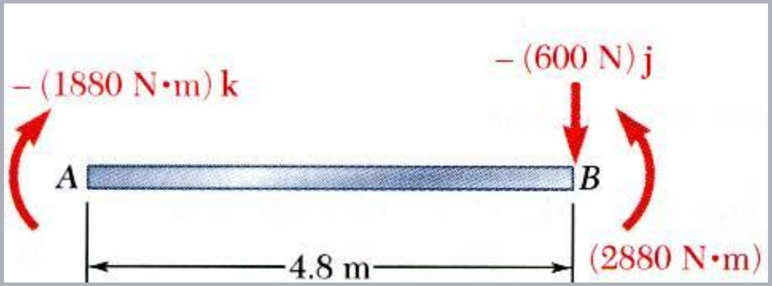
Sample Problem 3.8



b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

The force is unchanged by the movement of the force-couple system from *A* to *B*.

$$\vec{R} = -(600\ \text{N})\vec{j}$$

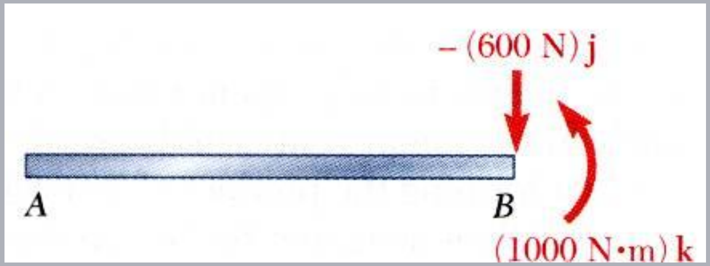


The couple at *B* is equal to the moment about *B* of the force-couple system found at *A*.

$$\vec{M}_B^R = \vec{M}_A^R + \vec{r}_{B/A} \times \vec{R}$$

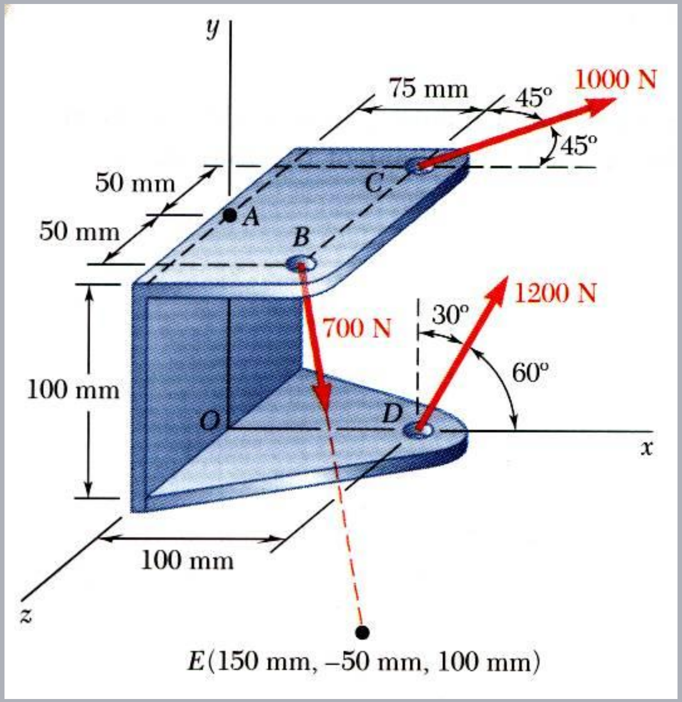
$$= -(1880\ \text{N}\cdot\text{m})\vec{k} + (-4.8\ \text{m})\vec{i} \times (-600\ \text{N})\vec{j}$$

$$= -(1880\ \text{N}\cdot\text{m})\vec{k} + (2880\ \text{N}\cdot\text{m})\vec{k}$$



$$\vec{M}_B^R = +(1000\ \text{N}\cdot\text{m})\vec{k}$$

Sample Problem 3.10



Three cables are attached to the bracket as shown. Replace the forces with an equivalent force-couple system at *A*.

SOLUTION:

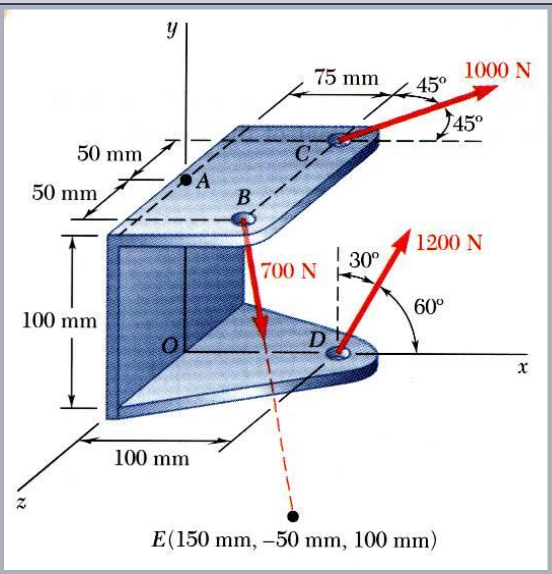
- Determine the relative position vectors for the points of application of the cable forces with respect to *A*.
- Resolve the forces into rectangular components.
- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$

- Compute the equivalent couple,

$$\vec{M}_A^R = \sum \left(\vec{r} \times \vec{F} \right)$$

Sample Problem 3.10



SOLUTION:

- Determine the relative position vectors with respect to *A*.

$$\vec{r}_{B/A} = 0.075\vec{i} + 0.050\vec{k}\ (\mathrm{m})$$

$$\vec{r}_{C/A} = 0.075\vec{i} - 0.050\vec{k}\ (\mathrm{m})$$

$$\vec{r}_{D/A} = 0.100\vec{i} - 0.100\vec{j}\ (\mathrm{m})$$

- Resolve the forces into rectangular components.

$$\vec{F}_B = (700\ \mathrm{N})\vec{\lambda}$$

$$\vec{\lambda} = \frac{\vec{r}_{E/B}}{r_{E/B}} = \frac{75\vec{i} - 150\vec{j} + 50\vec{k}}{175}$$

$$= 0.429\vec{i} - 0.857\vec{j} + 0.289\vec{k}$$

$$\vec{F}_B = 300\vec{i} - 600\vec{j} + 200\vec{k}\ (\mathrm{N})$$

$$\vec{F}_C = (1000\ \mathrm{N})(\cos 45\vec{i} - \cos 45\vec{j})$$

$$= 707\vec{i} - 707\vec{j}\ (\mathrm{N})$$

$$\vec{F}_D = (1200\ \mathrm{N})(\cos 60\vec{i} + \cos 30\vec{j})$$

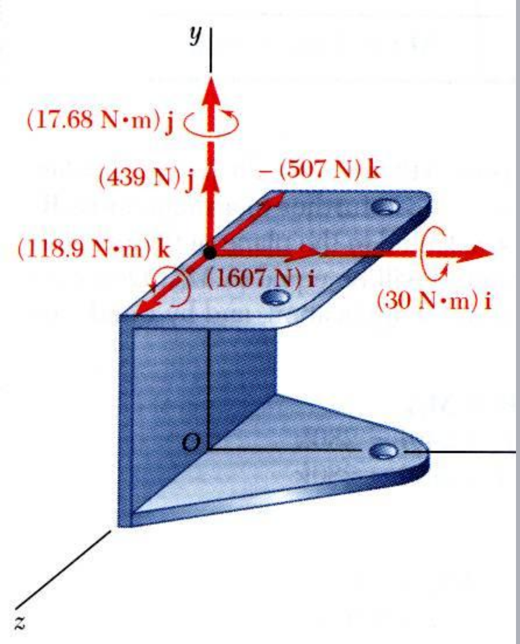
$$= 600\vec{i} + 1039\vec{j}\ (\mathrm{N})$$

Sample Problem 3.10

- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$
$$= (300 + 707 + 600)\vec{i}$$
$$+ (-600 + 1039)\vec{j}$$
$$+ (200 - 707)\vec{k}$$

$$\vec{R} = 1607\vec{i} + 439\vec{j} - 507\vec{k} \ \ (\mathrm{N})$$



- Compute the equivalent couple,

$$\vec{M}_A^R = \sum(\vec{r} \times \vec{F})$$

$$\vec{r}_{B/A} \times \vec{F}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\vec{i} - 45\vec{k}$$

$$\vec{r}_{C/A} \times \vec{F}_c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = 17.68\vec{j}$$

$$\vec{r}_{D/A} \times \vec{F}_D = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = 163.9\vec{k}$$

$$\vec{M}_A^R = 30\vec{i} + 17.68\vec{j} + 118.9\vec{k}$$