Total mark: 30. Closed book. Non-programmable calculators are allowed.

Last Name ____________ ***First Name*** ____________ ***Student Number*** ____________

Question 1. [2] Multiple Choice Questions(Please **CIRCLE** your answer)

(1) (2 Marks) A particular solution of the equation $y'' + 4y' + 4y = e^{-2x}$ is given by

A. Cx^3e^{-2x}

B. Cx^2e^{-2x}

C. Cxe^{-2x}

D. Ce^{-2x}

E. None of these

Solution: (1) B

Question 2. [8] Solve the IVP:

$$y'' - 4y' + 5y = 0,\ y(\pi) = 0,\ y'(\pi) = -e^{2\pi} \tag{1}$$

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Solution: The indicial equation is

$$r^2 - 4r + 5 = 0 \qquad \text{(1 mark)}$$

he roots of the indicial equation are

$$r_{1,2} = \frac{4 \pm 2i}{2} = 2 \pm i \qquad \text{(1 mark)}$$

so the general solution is

$$y = e^{2x}[c_1 \cos(x) + c_2 \sin(x)] \qquad \text{(2 marks)}$$

Since

$$y' = 2e^{2x}[c_1 \cos(x) + c_2 \sin(x)] + e^{2x}[-c_1 \sin(x) + c_2 \cos(x)]$$

and $y(\pi) = 0,\ y'(\pi) = -e^{2\pi}$, we have

$$\begin{aligned} -e^{2\pi}c_1 &= 0 \\ e^{2\pi}(-2c_1 - c_2) &= -e^{2\pi} \end{aligned} \qquad \text{(2 marks)}$$

so $c_1 = 0,\ c_2 = 1$. Then the solution to the initial-value problem is

$$y = e^{2x}\sin(x) \qquad \text{(2 marks)}$$

Question 3. [10] Solve the differential equation:

$$y'' - 2y' - 3y = 4e^{2x}. \tag{2}$$

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Solution: Solution 1 Two linearly-independent solutions to the complementary equation of (2) are

$$y_1(x) = e^{-x}, \;\; y_2(x) = e^{3x} \qquad \text{(2 Marks)}$$

Then, the Wronskian is

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} = 4e^{2x} \qquad \text{(2 Marks)}$$

and

$$\begin{aligned} W_1(x) &= \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} 0 & e^{3x} \\ 4e^{2x} & 3e^{3x} \end{vmatrix} = -4e^{5x} \\ W_2(x) &= \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix} = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & 4e^{2x} \end{vmatrix} = 4e^{x} \end{aligned} \qquad \text{(2 Marks)}$$

So, the particular solution is

$$\begin{aligned} y_p(x) =& y_1(x) \int \frac{W_1(x)}{W(x)} dx + y_2(x) \int \frac{W_2(x)}{W(x)} dx \\ =& e^{-x} \int \frac{-4e^{5x}}{4e^{2x}} dx + e^{3x} \int \frac{4e^{x}}{4e^{2x}} dx \\ =& e^{-x} \int -e^{3x} dx + e^{3x} \int e^{-x} dx \\ =& -\frac{4}{3} e^{2x} \end{aligned} \qquad \text{(2 Marks)}$$

Therefore, the general solution is

$$Y(x) = C_1 e^{-x} + C_2 e^{3x} - \frac{4}{3} e^{2x} \qquad \text{(2 Marks)}$$

Solution: Solution 2 Two linearly-independent solutions to the associated homogeneous equation of (2) are

$$y_1(x) = e^{-x}, \;\; y_2(x) = e^{3x} \qquad \text{(2 Marks)}$$

Since non-homogeneous term is $5e^{3x}$, we seek a particular solution of the form

$$y_p(x) = Ae^{2x}. \;\; \text{[1 Mark]} \tag{3}$$

Then,

$$\begin{aligned} y_p'(x) =& 2Ae^{2x} \\ y''_p(x) =& 4Ae^{2x} \;\; \text{[2 Marks]} \end{aligned} \tag{4}$$

Substitute both (3) and (4) into (2), we have

$$-3Ae^{2x} = 4e^{2x} \qquad \text{(2 Marks)}$$

So

$$-3A = 4 \;\; \text{[1 Mark]} \tag{5}$$

then

$$A = -\frac{4}{3}.$$

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A particular solution is therefore

$$y_p(x) = -\frac{4}{3}e^{2x} \qquad \text{(1 Mark)}$$

Therefore, the general solution is

$$Y(x) = C_1e^{-x} + C_2e^{3x} - \frac{4}{3}e^{2x} \qquad \text{(1 Mark)}$$

Question 4. [10] Solve the differential equation:

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}. \tag{6}$$

Solution: Two linearly-independent solutions to the complementary equation of (6) are

$$y_1(x) = e^{-2x}, \;\; y_2(x) = xe^{-2x} \qquad \text{(2 Marks)}$$

Then, the Wronskian is

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x} \qquad \text{(2 Marks)}$$

and

$$W_1(x) = \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} 0 & xe^{-2x} \\ \frac{e^{-2x}}{x^2} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = -\frac{e^{-4x}}{x}$$

$$W_2(x) = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix} = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{e^{-2x}}{x^2} \end{vmatrix} = \frac{e^{-4x}}{x^2} \qquad \text{(2 Marks)}$$

So, the particular solution is

$$\begin{aligned} y_p(x) =& y_1(x)\int \frac{W_1(x)}{W(x)}dx + y_2(x)\int \frac{W_2(x)}{W(x)}dx \\ =& e^{-2x}\int -\frac{1}{x}dx + xe^{-2x}\int \frac{1}{x^2}dx \\ =& -e^{-2x}\ln|x| - e^{-2x} \end{aligned} \qquad \text{(2 Marks)}$$

Therefore, the general solution is

$$Y(x) = C_1e^{-2x} + C_2xe^{-2x} - e^{-2x}\ln|x| - e^{-2x} \qquad \text{(2 Marks)}$$