Total mark: 30. Closed book. Non-programmable calculators are allowed.

Last Name ____________ First Name ____________ Student Number ____________

Question 1. [4 Marks] Multiple Choice Questions(Please **CIRCLE** your answer)

(1) (2 Marks) The equation $\frac{dy}{dx} = \frac{x^2y+y^3}{xy^2-x^2y^2}$ is

A). Separable B). Homogeneous C). Bernoulli D). Exact **E). None of these**

(2) (2 Marks) An integrating factor for the non-exact equation $(2y^2 + 3x)dx + 2xydy = 0$ is

A). y **B).** x C). e^y D). x^2 E). None of these

Solution: (1) E (2) B

Question 2. [6 Marks] Solve the differential equation

$$\frac{dy}{dx} = \frac{3x^2}{\cos y}$$

Solution: This equation is separable and we write it as

$$\cos y dy = 3x^2 dx \qquad \text{(3 marks)}$$

Integrate both sides, we have

$$\int \cos y dy = \int 3x^2 dx$$

$$\sin y = x^3 + C \qquad \text{(3 marks)}$$

That is the general solution.

Question 3. [10 Marks] Solve the differential equation

$$y' + 2xy = 4x^3 e^{-x^2}$$

Solution: This equation is first-order linear with $P(x) = 2x$ and $Q(x) = 4x^3e^{-x^2}$. An integrating factor is

$$I(x) = e^{\int 2x dx} = e^{x^2} \qquad \text{(4 marks)}$$

Multiplying both sides of the differential equation by $I(x)$, we have

$$e^{x^2}(y' + 2xy) = 4x^3 \qquad \text{(2 marks)}$$

or

$$\frac{d}{dx}[ye^{x^2}] = 4x^3$$

Integrating both sides, we get

$$ye^{x^2} = x^4 + C \qquad \text{(4 marks)}$$
$$y = x^4e^{-x^2} + Ce^{-x^2}$$

That is the general solution.

Question 4. [10 Marks] Solve the initial-value problem

$$(2xy^2 - 3)dx + (2x^2y + 4)dy = 0, \quad y(0) = 1$$

Solution: Solution 1 This equation is exact because

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy^2 - 3) = 4xy = \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2x^2y + 4) \qquad \text{(2 marks)}$$

The general solution $f(x, y) = C$ is given by

$$f(x, y) = \int P(x, y)dx = \int (2xy^2 - 3)dx = x^2y^2 - 3x + h(y) \qquad \text{(2 marks)}$$

Differentiate $f(x, y)$ with respect to y and compare the result with $Q(x, y)$

$$f_y(x, y) = 2x^2y + h'(y) = 2x^2y + 4$$

so $h'(y) = 4$. It follows that $h(y) = 4y$. Therefore

$$f(x, y) = x^2y^2 - 3x + 4y \qquad \text{(2 marks)}$$

The general solution is

$$x^2y^2 - 3x + 4y = C \qquad \text{(2 marks)}$$

Since $y(0) = 1$, we have

$$4 = C$$

So, the solution to the initial-value problem is

$$x^2y^2 - 3x + 4y = 4 \qquad \text{(2 marks)}$$

Solution: Solution 2 This equation is exact because

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy^2 - 3) = 4xy = \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2x^2y + 4) \qquad \text{(2 marks)}$$

The general solution $f(x, y) = C$ is given by

$$f(x, y) = \int Q(x, y)dy = \int (2x^2y + 4)dy = x^2y^2 + 4y + h(x) \qquad \text{(2 marks)}$$

Differentiate $f(x, y)$ with respect to x and compare the result with $P(x, y)$

$$f_x(x, y) = 2x^2y + h'(x) = 2xy^2 - 3$$

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so $h'(x) = -3$. It follows that $h(x) = -3x$. Therefore

$$f(x, y) = x^2y^2 - 3x + 4y$$ (2 marks)

The general solution is

$$x^2y^2 - 3x + 4y = C$$ (2 marks)

Since $y(0) = 1$, we have

$$4 = C$$

So, the solution to the initial-value problem is

$$x^2y^2 - 3x + 4y = 4$$ (2 marks)