MATH 1107E _ Review Period _ Examples solution.

Example 1:

a) $Ax = b$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \therefore x = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$$

b) $b = -4v_1 + 4v_2 + 3v_3$, where $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

c) The system is trivial

$\therefore$ the solution of $A x = \vec{0}$ is $\vec{0}$.

Example 2:

$G = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$ is not a subspace.

Fails under the addition condition.

$\Rightarrow$ Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in G$, $\vec{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \in G$

Then $\vec{u} + \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \notin G$.

$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$ is not a subspace.

Fails under the scalar multiplication condition

$\Rightarrow$ Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in H$ and $c = -1$.

Then $c\vec{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin H$.

$I = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \leq y-3 \right\}$ is not a subspace.

~~Let~~ $\vec{0} \notin I$

$\Rightarrow$ Let $x = 0$

then, $0 \leq y-3 \Rightarrow y \geq 3$, so cannot be 0.

$J = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x+y-z=0 \right\}$ is a subspace.

$= \left\{ \begin{bmatrix} x \\ y \\ 2x+y \end{bmatrix} \right\}$

1. Let $x, y = 0$, then $\vec{0} \in J$.

2. Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ 2x_1+y_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ 2x_2+y_2 \end{bmatrix} \in J$

Then, $\vec{u}+\vec{v} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ 2x_1+2x_2+y_1+y_2 \end{bmatrix} \in J$.

3. c is a scalar.

Then, $c\vec{u} = \begin{bmatrix} cx_1 \\ cy_1 \\ c(2x_1+y_1) \end{bmatrix} \in J$.

Example 3

a)

$$\text{Basis of Col } A = \left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

b)

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \Bigg\} \quad \begin{aligned} x_5 &= t_3 \\ x_4 &= t_2 \\ x_3 &= -2t_2 + 2t_3 \\ x_2 &= t_1 \\ x_1 &= 2t_1 + 2t_2 - 3t_3 \end{aligned}$$

$$\text{Null} A = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This set is clearly linearly independent and it spans the Null space.

$$\therefore \text{ Basis Null } A = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

c)

$$\text{Basis Row } A = \left\{ [1\ -2\ 0\ -1\ 3],\ [0\ 0\ 1\ 2\ -2] \right\}$$

d)

$$\left. \begin{aligned} \text{Rank} &= 2 \\ \text{Nullity} &= 3 \end{aligned} \right\} \quad \begin{aligned} n &= \text{Rank} + \text{Nullity}. \\ 5 &= 2 + 3 \end{aligned}$$

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Example 4

Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

$$T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1+x_2)+(y_1+y_2) \\ 3(x_1+x_2) \\ 5(x_1+x_2)-2(y_1+y_2) \end{bmatrix}$$

$$= \begin{bmatrix} (2x_1+y_1)+(2x_2+y_2) \\ 3x_1 + 3x_2 \\ (5x_1-2y_1)+(5x_1-2y_2) \end{bmatrix}$$

$$= T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$\therefore T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

2. Let c be a scalar.

$T(c\vec{u}) = cT(\vec{u})$

$$T\left(\begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}\right) = \begin{bmatrix} 2(cx_1)+cy_1 \\ 3(cx_1) \\ 5(cx_1)-2(cy_1) \end{bmatrix} = \begin{bmatrix} c(2x_1+y_1) \\ c(3x_1) \\ c(5x_1-2y_1) \end{bmatrix}$$

$$cT\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$\therefore T(c\vec{u}) = cT(\vec{u})$ and T is a linear transformation!

Example 5

$$A = \begin{bmatrix} 0 & 1 & -5 & 7 & 6 \\ -3 & 1 & -8 & 1 & 3 \end{bmatrix}$$

a) $\text{Range} = \left\{ \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

b) $$\left[\begin{array}{ccccc|c} 0 & 1 & -5 & 7 & 6 & 0 \\ -3 & 1 & -8 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & -5 & 7 & 6 & 0 \end{array}\right]$$

$$\left.\begin{aligned} x_5 &= t_3 \\ x_4 &= t_2 \\ x_3 &= t_1 \\ x_2 &= 5t_1 - 7t_2 - 6t_3 \\ x_1 &= -t_1 - 2t_2 - t_3 \end{aligned}\right\} \text{Kernel} = \left\{ \begin{bmatrix} -1 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -7 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

c) Nullity $\neq 0$ $\therefore$ Not injective

dim (codomain) = 2 = Rank of T $\therefore$ Surjective.

Example 6:

a) $\det\left(\begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}\right) = -2.$

b) $\text{Adj } A = \begin{pmatrix} -2 & 0 & -5 \\ 0 & 0 & 1 \\ -4 & 2 & -6 \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{-2}\begin{bmatrix} -2 & 0 & -5 \\ 0 & 0 & 1 \\ -4 & 2 & -6 \end{bmatrix}$

$\Rightarrow$ To check if you have the right inverse, see if $\boxed{A^{-1}A = I}$.

c) $\left(\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right)$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 5/2 \\ 0 & 1 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right)$

Example 7:

a)

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)(-1)^4 \begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left((4-\lambda)(1-\lambda) + 2\right) = 0$$

$$(1-\lambda)(\lambda-3)(\lambda-2) = 0$$

$$\therefore \lambda = 1, 3, 2$$

For $\lambda = 1$

$$A - I = 0$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$:

$A - 3I = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$:

$A - 2I = 0$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \vec{v}_3' = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \rightarrow \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

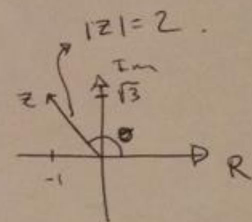
b) A is diagonalizable as there are three distinct eigenvectors.

Example 7 (continued).

c) - Geometric multiplicity is one for each eigen values.

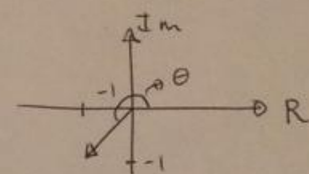
- Algebraic multiplicity is one for each eigen values.

Example 8:



a) $z = 2\,\mathrm{CIS}\left(\frac{2\pi}{3}\right)$

c) $g = -1 - i$



b) $z^{15} = 2^{15}\,\mathrm{CIS}\left(15 \cdot \frac{2\pi}{3}\right)$

d) $\bar{z} = -1 - i\sqrt{3}$

$\bar{g} = -1 + i$