Solutions of Linear Equations

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \quad \cdots \quad 0 \quad b] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Pivot (leading entry), REF and RREF

DEFINITION

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Linear Combination (Ax = b)

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{6}$$

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Linear Depedency and Homogenous System

DEFINITION

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{2}$$

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Example 1

Given the following system of equations:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

a) Solve *Ax=b*.

b) Find the corresponding linear combination.

c) Solve the homogeneous system.

Subspace, Basis and Dimension of a Subspace (and Space)

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

DEFINITION

A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

DEFINITION

The **dimension** of a nonzero subspace H, denoted by $\dim H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.[2]

Column Space, Null Space and Rank Theorem

DEFINITION

The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.

DEFINITION

The **null space** of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

DEFINITION

The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

THEOREM 14

The Rank Theorem

If a matrix A has n columns, then $\operatorname{rank} A + \dim \operatorname{Nul} A = n$.

Example 2

Which of the following are subspaces?

$$G = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$$

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$$

$$I = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \leq y - 3 \right\}$$

$$J = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + y - z = 0 \right\}$$

Example 3

Given the following system:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \approx \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find the basis of the Column space of A.
b) Find the basis of the Null Space of A.
c) Find the basis of the Row Space of A.
d) Find the Rank and Nullity of A.

Linear Transformation

DEFINITION

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

The **Kernel** of *T* is u in *V* such that *T*(u)=0.

The **Range** of *T* is the set of all vectors in *W* (in the form of *T*(x)) for some x in *V*.

T is **surjective** (onto) iff* the dimension of the codomain is equal to the Rank of *T*.

T is **injective** (one-to-one) iff* the nullity (dim(Kernel)) is 0.

T is **bijective** iff* it is both injective and surjective.

*iff means if and only if

Example 4

Given the transformation T from R^2 to R^3:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} 2x + y \\ 3x \\ 5x - 2y \end{bmatrix}\right)$$

Is T a linear transformation?

Example 5

Given the transformation T from R^5 to R^2:

$$T\left(\begin{bmatrix} x \\ y \\ z \\ \beta \\ \gamma \end{bmatrix}\right) = \left(\begin{bmatrix} 1y - 5z + 7\beta + 6\gamma \\ -3x + 1y - 8z + \beta + 3\gamma \end{bmatrix}\right)$$

a) Find the Range of T.
b) Find the Kernel of T.
c) Is T injective, surjective or bijective?

Finding the Inverse of a Matrix

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

THEOREM 8

An Inverse Formula

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\,\text{adj}\,A$$

THEOREM 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The Determinant of a Matrix

DEFINITION

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

THEOREM 2

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

THEOREM 3

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Finding the Adjoint of a Matrix

Steps for finding the adjugate:

1- Find every cofactor of A by computing the following formula:

$$C_{ji} = (-1)^{i+j} \det A_{ji}$$

2- Set the adjugate (adjoint) matrix (Note that the index of the cofactors C are switched, therefore the matrix is transposed):

$$Adj(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Some Properties of the Inverse

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

THEOREM 6

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

THEOREM 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example 6

Given the following system:

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

a) Find the determinant of A by row reduction.

b) Find the inverse of A using the inverse formula.

c) Find the inverse of A by solving A augmented with I.

Finding Eigenvalues and Eigenvectors

1- Compute the following formula:

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \qquad (1)$$

2- Find the **determinant** of the result. We know (1) is not invertible, so solve:

$$\det(A - \lambda I) = 0 \qquad (2)$$

3- The **eigenvalues** are simply the roots of the solution of (2).

4- To find the **eigenvectors**, back substitute one eigenvalue at a time in (1), and solve for each system.

Diagonalization

$$A = PDP^{-1}$$

THEOREM 5

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

THEOREM 2

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Example 7

Given the following matrix A.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

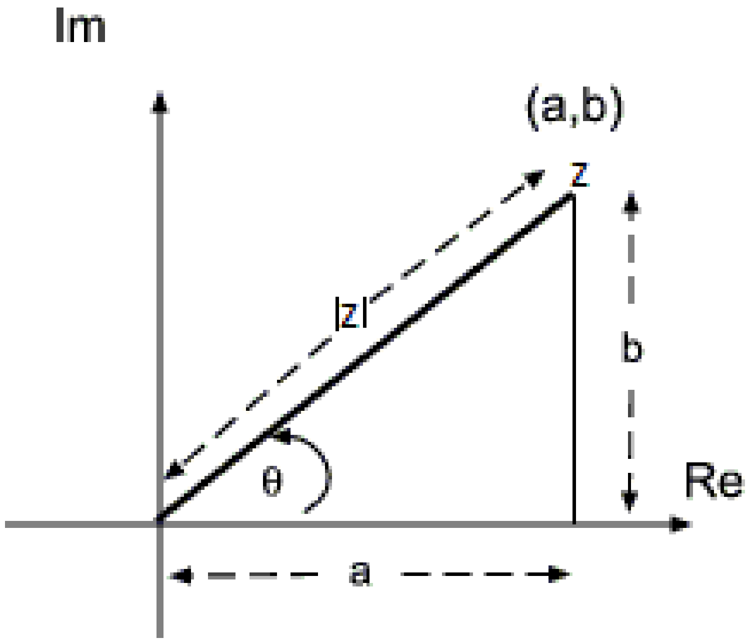
a) Find P and D such that $A=PDP^{-1}$
b) Is A diagonalizable?
c) What is the geometric and algebraic multiplicity?

Complex Numbers!

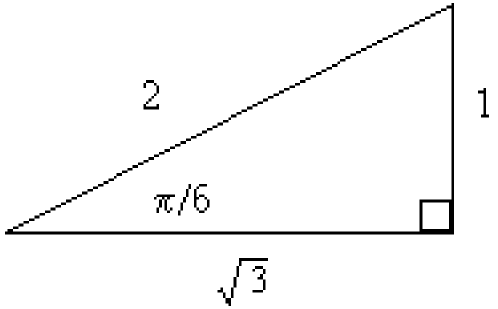
-- Standard Form: $z = a + bi$

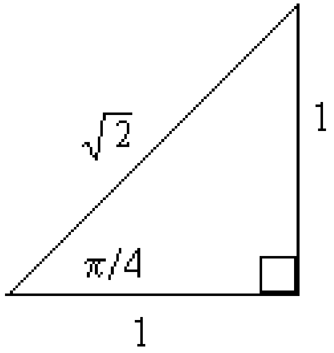
-- Polar Form: $z = |z|\ \cos\theta + \ |z|\sin\theta$

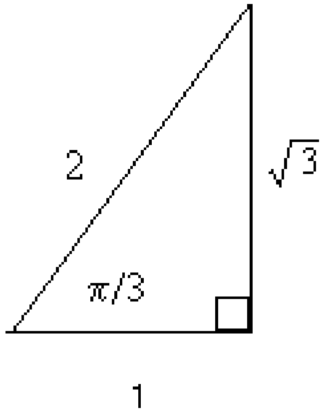
-- Complex Conjugate: The Imaginery part is negated and the Real part stays unchanged. The complex conjugate of z is denoted by $\bar{z}$.



-- Special Triangles:







Example 8

Consider the complex number *z* and *g*:

$$z = -1 + i\sqrt{3}$$

$$g = \sqrt{2}CIS\left(\frac{5\pi}{4}\right)$$

a) Change *z* in Polar Form.
b) Find z^{15}
c) Change *g* in Standard Form.
d) Find the complex conjugate of *z* and *g*.