MATH 1005 A **Name:**____________________

Sample Final **Student Number:**____________________

Recall that only **non-programmable** calculators are permitted. Books and/or notes are not permitted.

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1. [**6 marks**] Consider the first-order equation

$$\frac{x}{x^2+y^2+1} - 2x + \left(\frac{y}{x^2+y^2+1} + 2y\right)\frac{dy}{dx} = 0 \tag{1}$$

(a) Show that equation (1) is exact.

(b) Find the general solution to (1).

Solution: a) Since $P(x,y) = x/(x^2+y^2+1) - 2x$ and $Q(x,y) = y/(x^2+y^2+1) + 2y$, we have

$$P_x(x,y) = \frac{-2xy}{(x^2+y^2+1)^2} = Q_y(x,y) \text{ [2]}$$

so the equation is exact.

b) By exactness, there exists $f = f(x,y)$ such that $f_x = P$ and $f_y = Q$ [1]. Partially integrating with respect to x gives

$$f(x,y) = \int f_x(x,y)dx = \int \frac{x}{x^2+y^2+1} - 2xdx = \frac{1}{2}\ln(x^2+y^2+1) - x^2 + g(y) \text{ [1].}$$

But then

$$f_y(x,y) = Q(x,y) = \frac{y}{x^2+y^2+1} + g'(y) \quad \Rightarrow \quad g'(y) = 2y \text{ [1].}$$

Thus, $g(y) = y^2 + C$ and the general solution is given implicity by

$$\frac{1}{2}\ln(x^2+y^2+1) - x^2 + y^2 = C \text{ [1].}$$

2. [**9 marks**] Consider the second-order equation

$$y'' - 6y' + 9y = 4e^{3x}. \tag{2}$$

(a) Find the general solution to the corresponding homogenous equation.

(b) Find a particular solution to (2) using the method of undetermined coefficients.

(c) What is the general solution to (2)?

Solution: a) The indicial equation is $r^2 - 6r + 9 = 0$ which has a single repeated root $r = 3$ [1]. Letting $y_1(x) = e^{3x}$ and $y_2(x) = xe^{3x}$, the general solution to the homogenous equation is

$$y_h(x) = c_1 e^{3x} + c_2 x e^{3x} \text{ [2]}.$$

b) Since $g(x) = 4e^{3x}$, and both e^{3x} and xe^{3x} are present in y_h, we pick $y_p(x) = Ax^2e^{3x}$ [1]. Differentiating,

$$y_p' = 3Ax^2e^{3x} + 2Axe^{3x} \quad \Rightarrow \quad y_p'' = 9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x} \text{ [2]}.$$

Subbing into equation (2) and simplifying, we get

$$2Ae^{3x} = 4e^{3x} \quad \Rightarrow \quad A = 2 \text{ [1]}.$$

Hence, $y_p(x) = 2x^2e^{3x}$ [1].

c) The general solution is $y(x) = y_h(x) + y_p(x) = c_1e^{3x} + c_2xe^{3x} + 2x^2e^{3x}$ [1].

3. [**12 marks**] Consider the second-order equation

$$x^2y'' + 5xy' + 3y = 4xe^{x^2}, \quad x > 0. \tag{3}$$

(a) Find the general solution to the corresponding homogenous equation.

(b) Find a particular solution to (3) using variation of parameters.

(c) What is the general solution to (3)?

Solution: a) The homogenous equation is $x^2y'' + 5xy' + 3y = 0$, which is of Cauchy–Euler type with indicial equation

$$ar^2 + (b-a)r + c = 0 = r^2 + 4r + 3 = 0 \;\Rightarrow\; r = -3, -1 \;\; [2].$$

Letting $y_1(x) = x^{-3}$ and $y_2(x) = x^{-1}$, the general solution is $y_h(x) = c_1y_1(x) + c_2y_2(x)$ [2].

b) The Wronskian is

$$W(x) = \det\begin{pmatrix} x^{-3} & x^{-1} \\ -3x^{-4} & -x^{-2} \end{pmatrix} = -x^{-5} + 3x^{-5} = 2x^{-5} \;\; [2].$$

To find the appropriate non-homogenous term, we put the equation in standard form to see that $f(x) = 4x^{-1}e^{x^2}$ [1]. Then

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}dx = -\int 2x^3e^{x^2}dx \;\; [1].$$

Let $u = x^2$ and $dv = 2xe^{x^2}$ to get $du = 2xdx$ and $v = e^{x^2}$. Integration by parts gives

$$u_1(x) = -x^2e^{x^2} + \int 2xe^{x^2}dx = -x^2e^{x^2} + e^{x^2} \;\; [1].$$

Also,

$$u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}dx = \int 2xe^{x^2}dx = e^{x^2} \;\; [1].$$

Putting things together, $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ [1].

c) The general solution to (3) is

$$y(x) = c_1x^{-3} + c_2x^{-1} + (1-x^2)e^{x^2}x^{-3} + e^{x^2}x^{-1} = c_1x^{-3} + c_2x^{-1} + e^{x^2}x^{-3} \;\; [1].$$

4. [**6 marks**] Find a fundamental matrix for the linear system

$$\mathbf{x}' = A\mathbf{x}; \quad A = \begin{pmatrix} 3 & 1 \\ -4 & -2 \end{pmatrix}.$$

Solution: Eigenvalues of A satisfy

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -1 \\ 4 & \lambda + 2 \end{vmatrix} = (\lambda - 3)(\lambda + 2) + 4 = (\lambda - 2)(\lambda + 1),$$

so $\lambda = -1, 2$ [2]. The eigenvectors for $\lambda = -1$ satisfy $(-1I - A)\mathbf{v} = \mathbf{0}$, or

$$\begin{pmatrix} -4 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies $4a = -b$, so picking $a = 1$ we get

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \quad [1].$$

The eigenvectors for $\lambda = 2$ satisfy $(2I - A)\mathbf{v} = \mathbf{0}$, or

$$\begin{pmatrix} -1 & -1 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies $-a = b$, so picking $a = 1$ we get

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad [1].$$

A fundamental matrix is therefore

$$X(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -4e^{-t} & -e^{2t} \end{pmatrix} \quad [2].$$

5. [**6 marks**] Consider the series

$$\sum_{n=1}^{\infty} \frac{\ln(n)^3}{n^2}. \tag{4}$$

(a) Show that the sequence $a_n = \frac{\ln(n)^3}{\sqrt{n}}$, $n \geq 1$, converges to 0 (note that $a_n \neq \frac{\ln(n)^3}{n^2}$).

(b) Use part (a) to show that the series (4) converges.

Solution: a) By repeated application of L'Hôpital's rule, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(x)^3}{\sqrt{x}} &= \lim_{x\to\infty} \frac{3\ln(x)^2}{2^{-1}\sqrt{x}} \\ &= \lim_{x\to\infty} \frac{3\cdot 2\ln(x)}{2^{-2}\sqrt{x}} \\ &= \lim_{x\to\infty} \frac{3\cdot 2}{2^{-3}\sqrt{x}} \\ &= 0 \ [2].\end{aligned}$$

Thus, $\lim_{n\to\infty} a_n = 0$ [1].

b) Since $(a_n)_{n=1}^{\infty}$ converges, it is bounded above by some $M > 0$ [1], that is, $a_n \leq M$ for all $n \geq 1$. Hence

$$\frac{\ln(n)^3}{n^2} = \frac{\ln(n)^3}{\sqrt{n}} \frac{1}{n^{3/2}} \leq \frac{M}{n^{3/2}}, \quad n \geq 1 \ [1].$$

Since $\sum_{n=1}^{\infty} \frac{M}{n^{3/2}}$ converges the p-test, it follows by the comparison test that (4) converges [1].

6. [**7 marks**] Consider the function $f(x) = \cos(\sqrt{x})$.

(a) Compute the Taylor series for $f(x)$ centered at $a = 0$. What is its radius of convergence?

(b) Approximate the integral

$$\int_0^1 f(x)dx$$

using a second degree Taylor approximation. Show that your answer is correct to 3 decimal places.

Solution: a) Using the Taylor series for $\cos(x)$, we easily obtain

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{(2k)!}, \quad x \in \mathbb{R} \ [1].$$

Hence, the radius of convergence is $R = \infty$ [1].

b) Using a second order approximation, we get

$$\begin{aligned}\int_0^1 f(x)dx &\approx \sum_{k=0}^{2} \frac{(-1)^k}{(2k)!} \int_0^1 x^k dx \\ &= \sum_{k=0}^{2} \frac{(-1)^k}{(2k)!} \left(\frac{x^{k+1}}{k+1} \right)_0^1 \\ &= \sum_{k=0}^{2} \frac{(-1)^k}{(2k)!(k+1)} \\ &= 1 - \frac{1}{4} + \frac{1}{4! \cdot 3} \\ &= \frac{55}{72} \ [3].\end{aligned}$$

Since the remainder is given by an alternating series, we know that

$$|R_2| < b_3 = \frac{1}{6! \cdot 4} = 0.0003472222 < 0.001 \ [2],$$

so our approximation is correct to 3 decimal places.

7. [**8 marks**] Consider the second-order equation

$$y'' + xy' - 2y = 0. \tag{5}$$

(a) Find the coefficient recursion relation for the general series solution about $x_0 = 0$.

(b) Solve the recursion relation to obtain the general solution to (5).

Solution: a) Since the $x_0 = 0$ is an ordinary point, we may represent y as $\sum_{n=0}^{\infty} c_n x^n$. Subbing this into the equation gives

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} 2c_n x^n = 0 \text{ [1]}.$$

Shifting the indicies and simplifying, we get

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + (n-2)c_n]x^n = 0 \text{ [1]}.$$

The recursion relation is therefore

$$c_{n+2} = \frac{-(n-2)}{(n+2)(n+1)} c_n \text{ [1]}.$$

b) Plugging in the first few cases for n even, we see that $c_2 = c_0$, and $c_{2k} = 0$ for all $k \geq 2$ [1]. For n odd, after a few steps (show your work) one sees that

$$c_{2k+1} = \frac{(-1)^{k-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2k-3)}{(2k+1)!} c_1, \quad k \geq 0 \text{ [3]}.$$

The general solution is therefore

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=1}^{\infty} c_{2k+1} x^{2k+1} \\ &= c_0(1 + x^2) + c_1 \sum_{k=0}^{\infty} \frac{(-1)^{k-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2k-3)}{(2k+1)!} x^{2k+1} \text{ [1]}. \end{aligned}$$