Physics 3K03

Day Class Prof. R. K. Bhaduri

Duration of Examination: 3 Hours

McMaster University Final Examination Dec. 06, 2006

Numerical value of constants:

(SI units) $\hbar = 1.05 \times 10^{-34}$ Js; $k_B = 1.38 \times 10^{-23}$ J/K; $c = 3 \times 10^8$m/s ; M for He(4) $= 4 \times 1.67 \times 10^{-27}$kg.

(in nuclear units) $\hbar c = 197$ MeV fm; 1 fm $= 10^{-13}$ cm; 1 MeV$= 10^6$ eV; Mc^2 for He(4)$=$ 4×940 MeV; $k_B = 0.86 \times 10^{-4}$ eV/K.

NOTE: First read the paper, and then **answer all FIVE questions**. All work submitted will be graded. Candidates may bring any Pocket calculator. You may need the vlaues of the constants above for numerical estimation.

This examination paper includes 5 questions on 2 pages. You are responsible for ensuring that your copy of the paper is complete. Please bring any discrepancy to the attention of an invigilator. This notice is required by resolution of the Senate.

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1. (a) Consider a classical monoatomic perfect gas with N particles in a volume V. Show that the entropy per particle is given by

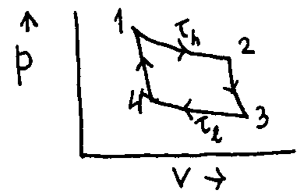
$$\frac{S}{N} = k_B \left[-\ln(n\lambda^3) + \frac{5}{2} \right] ,$$

where $n = N/V$, $\lambda = \sqrt{2\pi\hbar^2/(M\tau)}$, and $\tau = k_B T$. You may assume that the one-particle canonical partition function is $Z_1 = (V/\lambda^3)$. Use Stirling's approximation for large N : $\ln N! = N \ln N - N$.

(b) Estimate the numerical value of $(n\lambda^3)$ for a He(4) atom at room temperature, $T = 300K$. Take $n = 10^{22}/cm^3$.

(c) What does the Sackur-Tetrode formula give for S/N at $T = 0$? What do you expect the correct value of S/N to be at $T = 0$? Comment on the applicability of the formula above.

(2) Consider the Carnot cycle with a perfect gas. The $p-V$ graph is shown in the diagram below.



The gas is expanded at a constant temperature T_h from $1 \to 2$. It is further expanded isentropically from $2 \to 3$, the temperature falling to T_l ($T_l < T_h$). It is then compressed isothermally at T_l from $3 \to 4$, and isentropically back to 1 from 4.

(a) Draw the Carnot cycle in the $S-T$ plane, with S as the vertical and T the horizontal axes, marking the points 1 to 4 clearly. You may use the Sackur-Tetrode formula of Question (1) to justify the equivalence of your $S-T$ plot to the $p-V$ one given above.

(b) Calculate the efficiency of the Carnot cycle, which is the ratio of the net work done/cycle to the heat in-take at T_h from the reservoir.

(3) Gibbs free energy is defined as $G = U - TS + PV$. In general, $G = \mu N$, where μ is the chemical potential, and N the number of particles in the volume V.

(a) Consider a photon gas in a large volume V at temperature T. Assuming $PV = \frac{1}{3}U$, show that

(i) $C_V = 3S$, and (ii) S is proportional to T^3.

(b) Noting that S is also proportional to the volume V, and S/k_B is dimensionless, write down an equation for S/k_B from dimensional considerations. (You have the constants $\hbar$, c and k_B at your disposal). Hence estimate S/k_B numerically for $V = 1\ cm^3$, and $T = 300K$.

(4) (a) Consider N spin-1/2 fermions in a volume V at $T = 0$. Show that the magnitude of the Fermi momentum p_F is given by $\hbar(3\pi^2 n)^{1/3}$, where $n = N/V$.

(b) Take a simple model of a charge-neutral neutron star with neutrons, protons, and electrons in a dynamic equilibrium. At very high densities, these are all extremely relativistic, so their rest masses may be neglected. Also, the Fermi energy $E_F >> \tau$, so we may neglect the temperature dependence of the chemical potential. At such high densities, electron capture by a proton is feasible, resulting in the nuclear reaction $\mathbf{p} + \mathbf{e}^- \rightleftarrows \mathbf{n} + \nu_\mathbf{e}$, where p, n, e^- and ν_e are the symbols for a proton, a neutron, an electron, and an electron-neutrino respectively. The neutrinos escape immediately. Using the results from (a), find the ratio of the number of protons to neutrons in the star.

(5) Two-orbital Bose system : Consider a system of N bosons of spin zero occupying orbitals with single-particle energies at 0 and ϵ. The chemical potential is μ and the temperature $T = \tau/k_B$. Find τ such that the thermal average population of the lowest orbital is twice the population of the orbital at energy ϵ. Assume $N >> 1$, and make whatever approximations seem reasonable.

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