Assignment #3 Solution

1. Consider the DC motor control system shown in Fig. 4.25 (a).

(a) What is the signal picked off at point *A* in the block diagram of Fig. 4.25 (a), motor speed or position? Name the sensor used to implement the inner feedback loop, as well as the sensor that implements the outer feedback loop.

(b) Fig. 4.25(b) and Fig. 4.25(a) are equivalent, meaning they give the same transfer function between θ_r and θ. Find values for K' and $k_t^{'}$ in block diagram Fig. 4.25(b).

(c) Determine the system type with respect to tracking θ_r and compute the error constant in terms of parameters K' and $k_t^{'}$.

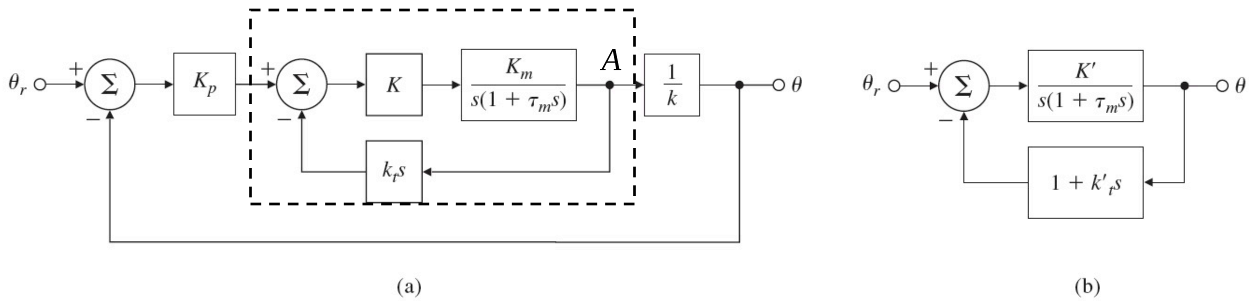


Fig. 4.25

Solution:

(a) Signals picked off at point *A* would be motor position. The inner feedback is has an "s" term, meaning differentiation. Therefore the inner feedback loop must be measuring the motor speed, a tachometer can be used. The outer feedback loop measures motor position factored by a constant 1/k, a potentiometer or an optical encoder can be used.

(b) The inner loop as indicated by dashed line box in Fig. 4.25(a) can be simplified by a transfer function as

$$G(s) = \frac{KK_m / s(1+\tau_m s)}{1+KK_m k_t /(1+\tau_m s)} = \frac{KK_m}{\tau_m s^2 + (1+KK_m k_t)s}$$

Then, the closed loop transfer function in Fig. 4.25(a) is

$$T_a(s) = \frac{K_p G(s)/k}{1+K_p G(s)/k} = \frac{\dfrac{K_p}{k}\dfrac{KK_m}{\tau_m s^2 + (1+KK_m k_t)s}}{1+\dfrac{K_p}{k}\dfrac{KK_m}{\tau_m s^2 + (1+KK_m k_t)s}} = \frac{KK_p K_m}{k\tau_m s^2 + k(1+KK_m k_t)s + KK_p K_m} \qquad (1)$$

$$= \frac{KK_p K_m / k}{\tau_m s^2 + (1+KK_m k_t)s + KK_p K_m / k}$$

On the other hand, the closed loop transfer function of Fig. 4.25 (b) is

$$T_b(s) = \frac{K' / s(1+\tau_m s)}{1+K'(1+k'_t\, s)/s(1+\tau_m s)} = \frac{K'}{\tau_m s^2 + (1+K' k'_t)s + K'} \qquad (2)$$

Equate (1) and (2), we conclude that:

$$K' = \frac{KK_p K_m}{k}$$

$$k'_t = \frac{kk_t}{K_p}$$

(c) From Fig.4.25.(a) we see that there is an integrator (1/s term) in a unity feedback, so we suspect that the system is Type 1. To check, use a ramp reference $\Xi_r(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} s\Xi_r(s)(1 - T_a(s)) = \lim_{s\to 0} s\left(\frac{1}{s^2}\right)\frac{1}{1 + K_p G(s)/k} = \lim_{s\to 0}\frac{1}{s + \frac{K_p}{k}\frac{KK_m}{\tau_m s + (1 + KK_m k_t)}} = \frac{1}{KK_p K_m / k(1 + KK_m k_t)}$$

Therefore, velocity error constant is: $K_v = \frac{KK_p K_m}{k(1 + KK_m k_t)}$

2. Consider the system shown in Fig. 4.27, which represents control of the angle of a pendulum that has no damping.
(a). What condition must $D(s)$ satisfy so that the system can track a ramp reference input with constant steady-state error?
(b). For a transfer function $D(s)$ that stabilizes the system and satisfies the condition in part (a), find the class of disturbances $w(t)$ that the system can reject with zero steady-state error.

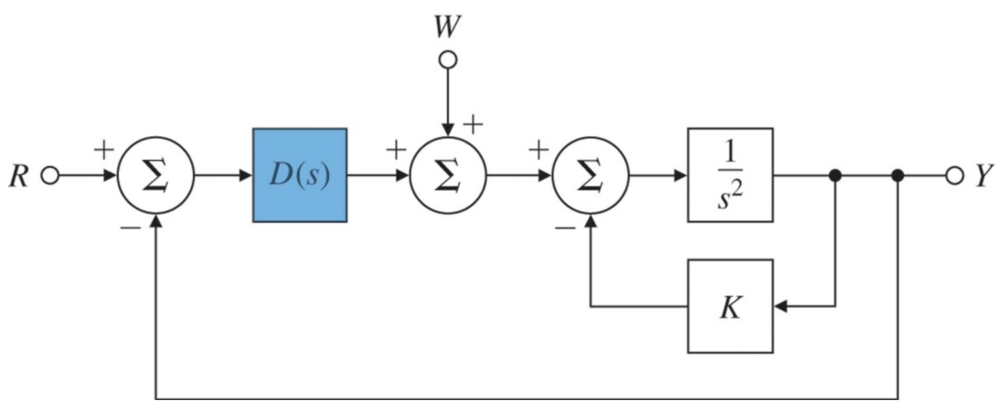


Fig. 4.27

Solution:
(a). The closed loop transfer function for reference tracking is:

$$T_R(s) = \frac{D(s)G(s)}{1 + D(s)G(s)} \text{ where } G(s) = \frac{\frac{1}{s^2}}{1 + K\frac{1}{s^2}} = \frac{1}{s^2 + K}$$

This system has a unity feedback, we know that an integrator ($\frac{1}{s}$ term) in the loop transfer function $D(s)G(s)$ is needed to produce a Type 1 closed-loop system. Since there is no $\frac{1}{s}$ term in $G(s)$, we must have an integrator in $D(s)$, that is, an $\frac{1}{s}$ term in $D(s)$. Hence, a PI control is likely the choice.
(b) The closed-loop transfer function from disturbance to the output is

$$T_w(s) = \frac{G(s)}{1 + D(s)G(s)}$$

Steady state error for disturbance rejection is:

$$e_{ss} = \lim_{s\to 0} s(0 - Y) = -\lim_{s\to 0} sT_w(s)W = -\lim_{s\to 0} s\frac{G(s)}{1 + D(s)G(s)}W$$

Start testing with a step disturbance:

$$e_{ss} = -\lim_{s\to 0} s\left(\frac{1}{s}\right)\frac{G(s)}{1+D(s)G(s)} = -\lim_{s\to 0}\frac{\frac{1}{s^2+K}}{1+D(s)\frac{1}{s^2+K}} = -\lim_{s\to 0}\frac{1}{s^2+K+D(s)}$$

Since $D(s)$ has a pole at origin (PI control: $D(s) = k_p + \frac{k_I}{s}$), $D(0) = \infty$, the above expression indicates that $e_{ss} = 0$. Therefore, the system can reject a step disturbance with zero steady state error. With a ramp disturbance,

$$e_{ss} = -\lim_{s\to 0} s\left(\frac{1}{s^2}\right)\frac{G(s)}{1+D(s)G(s)} = -\lim_{s\to 0}\frac{1}{s^3+Ks+sD(s)} = -\frac{1}{k_I} \neq 0$$

3. Consider the second-order system

$$G(s) = \frac{1}{s^2 + 2\zeta\, s + 1}$$

We would like to add a transfer function of the form $D(s) = \frac{K(s+a)}{(s+b)}$ in series with $G(s)$ in a unity-feedback structure.

(a) Ignoring stability for the moment, what are the constraints on K, a, and b so that system is Type 1?

(b) What are the constraints placed on K, a, and b so that the system is stable and Type 1?

(c) What are the constraints on a and b so that the system is Type 1 and remains stable for every positive value for K?

Solution:

a). Consider the loop gain $L(s) = D(s)G(s) = \frac{K(s+a)}{(s+b)(s^2+2\varsigma\, s+1)}$. For the system to be Type 1, there must be one integrator in $L(s)$. Since $G(s)$ doesn't have an integrator already, there must be one integrator in the controller, meaning $b=0$ and $a \neq 0$. Use $D(s) = \frac{K(s+a)}{s}$ to check:

For Type 1 steady-state error tracking, use ramp reference to test, that is, $R(s) = \frac{1}{s^2}$.

$$e_{ss} = \lim_{s\to 0} s(R(s)-Y(s)) = \lim_{s\to 0} sR(s)(1-T(s)) = \lim_{s\to 0} sR(s)\left(1-\frac{D(s)G(s)}{1+D(s)G(s)}\right)$$

$$= \lim_{s\to 0} sR(s)\left(\frac{1}{1+D(s)G(s)}\right) = \lim_{s\to 0} s\frac{1}{s^2}\left(\frac{1}{1+\frac{K(s+a)}{s}\frac{1}{s^2+2\varsigma s+1}}\right) = \frac{1}{Ka} \neq 0 \Rightarrow K_v = Ka$$

b) Check the characteristics equation:

$$CE: 1+D(s)G(s) = 0 \quad \Rightarrow \quad s^3 + 2\varsigma s^2 + (1+K)s + aK = 0$$

Use Routh stability method:

s^3	1	$1+K$
s^2	2ς	aK
s^1	$\frac{2\varsigma(1+K)-aK}{2\varsigma}$	0
s^0	aK	

For a stable closed-loop system, all elements in the first column must be positive (since it started from being positive and no sign change is allowed). Assuming $\varsigma > 0$ as it belongs to the plant model $G(s)$ which we don't have control on, the following constraints must be imposed on the controller parameter a, b, and K:

$$2\varsigma(1+K) - aK > 0 \quad \text{AND} \quad aK > 0$$

c) For any positive values of K, if the above conditions are satisfied, then the system remains stable (and Type 1). Constraints must be imposed on a as follows:

$$0 < a < \frac{2\varsigma(1+K)}{K}$$

4. A compensated motor position control system is shown in Fig. 4.29. Assume that the sensor dynamics is $H(s)$=1.

(a) Can the system track a step reference input r with zero steady-state error? If yes, give the value of the velocity error constant.

(b) Can the system reject a step disturbance w with zero steady-state error? If yes, give the value of the velocity error constant.

(c). In some instances there are dynamics in the sensor. Repeat parts (a) and (b) for $H(s) = \dfrac{20}{(s+20)}$ and compare the corresponding velocity error constants.

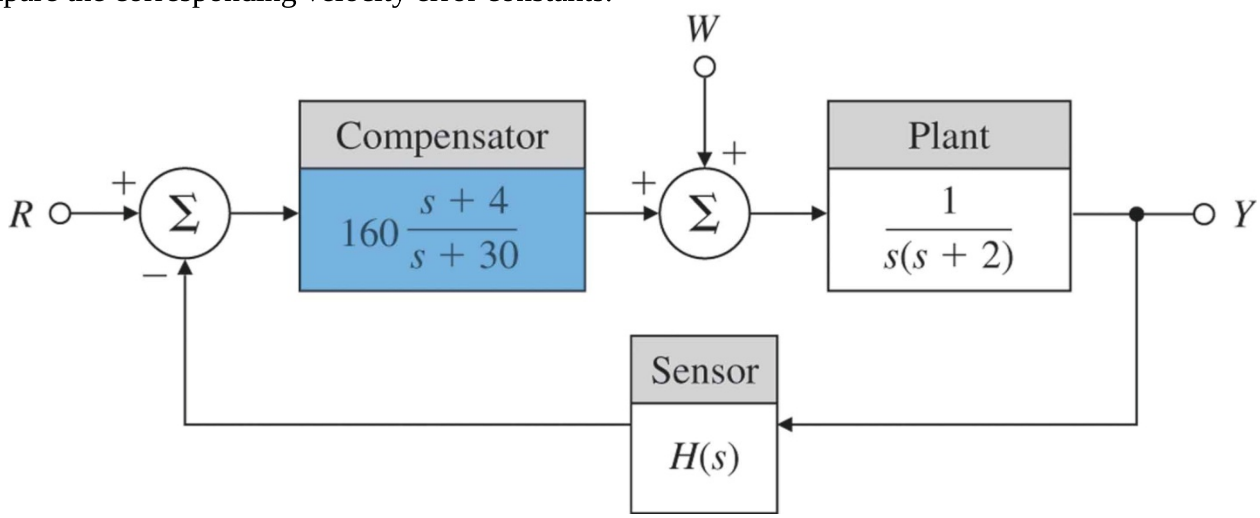


Fig. 4.29.

Solution:

(a) For a step reference input $R(s) = \dfrac{1}{s}$:

$$e_{ss} = \lim_{s\to 0} s\big(R(s) - Y(s)\big) = \lim_{s\to 0} sR(s)\left(1 - \frac{Y(s)}{R(s)}\right) = \lim_{s\to 0} sR(s)\big(1 - T(s)\big) = \lim_{s\to 0} sR(s)\left(1 - \frac{D(s)G(s)}{1 + D(s)G(s)H(s)}\right)$$

$$= \lim_{s\to 0} s\left(\frac{1}{s}\right)\frac{1 + D(s)G(s)H(s) - D(s)G(s)}{1 + D(s)G(s)H(s)} = \lim_{s\to 0} \frac{1}{1 + 160\left(\dfrac{s+4}{s+30}\right)\left(\dfrac{1}{s(s+2)}\right)\cdot 1} = 0$$

Therefore, the system can track a step reference with no error. To find the velocity error constant, $R(s) = \dfrac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} s\left(\frac{1}{s^2}\right)\frac{1+D(s)G(s)H(s)-D(s)G(s)}{1+D(s)G(s)H(s)} = \lim_{s\to 0}\frac{1}{s}\cdot\frac{1}{1+160\left(\frac{s+4}{s+30}\right)\left(\frac{1}{s(s+2)}\right)\cdot 1}$$

$$= \lim_{s\to 0}\frac{1}{s+160\left(\frac{s+4}{s+30}\right)\frac{1}{(s+2)}} = \frac{1}{\frac{160(4)}{30(2)}} = \frac{1}{10.64} \Rightarrow K_v = 10.64$$

(b) For a step disturbance $W(s) = \frac{1}{s}$,

$$e_{ss} = \lim_{s\to 0} s(0-Y) = -\lim_{s\to 0} sW(s)T_w(s) = -\lim_{s\to 0} s\left(\frac{1}{s}\right)\frac{G(s)}{1+D(s)G(s)H(s)} = -\lim_{s\to 0}\frac{\frac{1}{s(s+2)}}{1+160\left(\frac{s+4}{s+30}\right)\frac{1}{s(s+2)}}$$

$$= -\lim_{s\to 0}\frac{1}{s(s+2)+160\left(\frac{s+4}{s+30}\right)} = -\lim_{s\to 0}\frac{30}{160(4)} = -0.047$$

Therefore, the system cannot reject a step disturbance with zero steady-state error. There is no velocity error constant.

(c). With $H(s) = \frac{20}{s+20}$, for reference tracking:

$$e_{ss} = \lim_{s\to 0} sR(s)\frac{1+D(s)G(s)H(s)-D(s)G(s)}{1+D(s)G(s)H(s)}$$

$$= \lim_{s\to 0} sR(s)\cdot\frac{1+160\left(\frac{s+4}{s+30}\right)\left(\frac{1}{s(s+2)}\right)\left(\frac{20}{s+20}\right)-160\left(\frac{s+4}{s+30}\right)\left(\frac{1}{s(s+2)}\right)}{1+160\left(\frac{s+4}{s+30}\right)\left(\frac{1}{s(s+2)}\right)\left(\frac{20}{s+20}\right)}$$

$$= \lim_{s\to 0} sR(s)\cdot\frac{s(s+2)(s+20)(s+30)-160(s+4)s}{s(s+2)(s+20)(s+30)+160(s+4)\cdot 20}$$

with $R(s) = \frac{1}{s}$, $e_{ss} = 0$

with $R(s) = \frac{1}{s^2}$, $e_{SS} = \frac{2(20)(30)-160(4)}{160(4)(20)} = \frac{1}{22.86} \Rightarrow K_V = 22.86$ Still Type 1 for reference tracking with however, a different velocity error constant than from the previous case;

For disturbance rejection:

$$e_{ss} = -\lim_{s\to 0} sW(s)\frac{G(s)}{1+D(s)G(s)H(s)} = -\lim_{s\to 0} sW(s)\frac{\frac{1}{s(s+2)}}{1+160\left(\frac{s+4}{s+30}\right)\frac{1}{s(s+2)}\left(\frac{20}{s+20}\right)}$$

$$= -\lim_{s\to 0} sW(s)\frac{(s+20)(s+30)}{s(s+2)(s+20)(s+30)+160(20)(s+4)}$$

with $W(s) = \frac{1}{s}$, $e_{ss} = -\frac{20(30)}{160(20)(4)} = -\frac{1}{1+20.33} \Rightarrow K_p = 20.33$ Again Type 0 for disturbance rejection.

5. The feedback control system shown in Fig. 4.47 is to be designed to satisfy the following specifications:

(1) steady-state error of less than 10% to a ramp reference input,
(2) maximum overshoot for a unit-step input of less than 5%, and
(3) 1% settling time of less than 3 sec.

(a) Compute the closed-loop transfer function.
(b) Sketch the region in the complex plane where the closed-loop poles may lie.
(c) What does specification (1) imply about the possible values of *A*?
(d) What does specification (3) imply about the closed-loop poles?
(e) Find the error due to a unit-ramp input in terms of *A* and k_t.

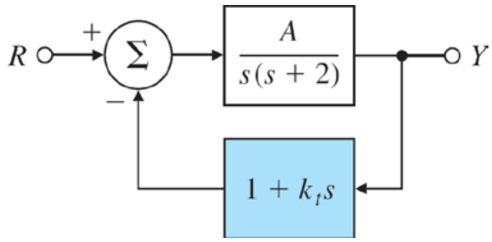


Fig. 4.47

Solution:

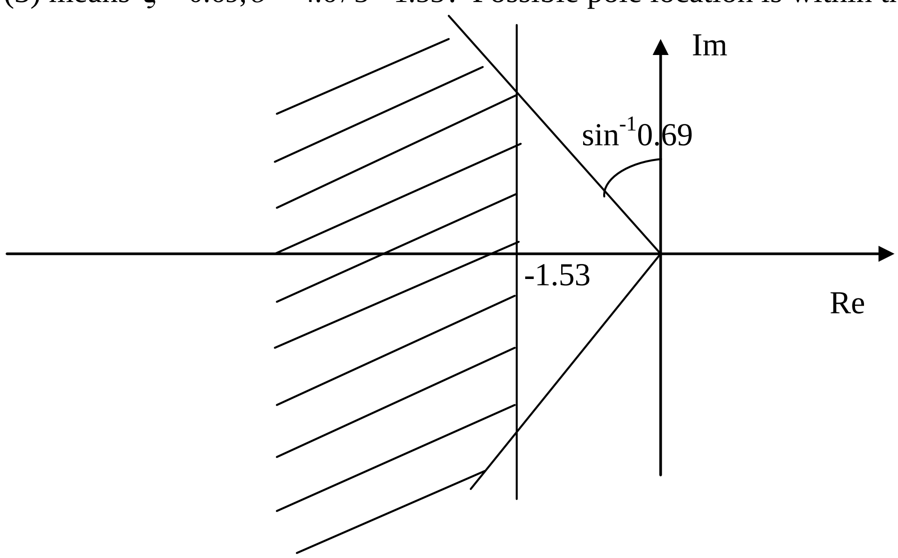
(a) $$T(s)=\frac{Y(s)}{R(s)}=\frac{\frac{A}{s(s+2)}}{1+(1+k_t s)\frac{A}{s(s+2)}}=\frac{A}{s(s+2)+A(1+k_t s)}=\frac{A}{s^2+(2+Ak_t)s+A}$$

(b)

$$\frac{A}{s^2+(2+Ak_t)s+A}=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\Rightarrow \omega_n=\sqrt{A},\ \xi=\frac{2+Ak_t}{2\sqrt{A}},\ \sigma=\frac{2+Ak_t}{2}>1$$

Requirements 2) and (3) means $\xi=0.69, \sigma=4.6/3=1.53$. Possible pole location is within the shaded area



(c) $e_{ss} = \lim_{s\to 0} s\left(\frac{1}{s^2}\right)(1-T(s)) = \left(\frac{s+2+Ak_t}{s^2+(2+Ak_t)s+A}\right)\Bigg|_{s=0} = \frac{2+Ak_t}{A} \le 0.1 \Rightarrow 2+Ak_t \le 0.1A$

(d) $\sigma = \frac{2+Ak_t}{2} > 1.53$

(e) This is a trivial question since steady-state error to a unit ramp input is already found in c).

Something worth thinking about: If one wants to figure out a range for A and k_t that satisfy all three requirements, we can combine $\xi = \frac{2+Ak_t}{2\sqrt{A}} > 0.69$ with the other two inequality relations in c) and d). One came to the conclusion:

$$\begin{matrix} 3.06 < 2+Ak_t < 0.1A \Rightarrow 0.36 < 0.1A \Rightarrow A > 30.6 \\ 1.4\sqrt{A} < 2+Ak_t < 0.1A \Rightarrow 1.4\sqrt{A} < 0.1A \Rightarrow A > 200 \end{matrix} \Rightarrow A > 200 \text{ and } 1.4\sqrt{A} < 2+Ak_t < 0.1A.$$

In fact, $k_t < 0.1$ is necessary as well from the e_{ss} requirement c) although it is not sufficient, but this will give us a smaller range to pick k_t and better chance to make things work. For design, one can pick a combination of A (>200) and k_t (<0.1), and then check to make sure that $1.4\sqrt{A} < 2+Ak_t < 0.1A$ is satisfied. For example, pick A=400, $k_t = 0.08$ then $e_{ss} = 0.085 < 0.1$; $\sigma = 17 > 1.53$; $\xi = 0.85 > 0.69$, meaning all three requirements are satisfied.

6. You are given the system shown in Fig. 4.35, where the feedback gain β is subject to variations. You are to design a controller for this system so that the output $y(t)$ accurately tracks the reference input $r(t)$.

(a) Let β=1 . You are given the following three options for the controller $D_i(s)$:

$$D_1(s) = k_p, \quad D_2(s) = \frac{k_p s + k_I}{s}, \quad D_3(s) = \frac{k_p s^2 + k_I s + k_2}{s^2}$$

Choose the controller (including particular values for the controller constants) that will result in a Type 1 system with a steady-state error to a unit reference ramp of less than $\frac{1}{10}$.

(b) Next, suppose that there is some attenuation in the feedback path that is modeled by β=0.9. Find the steady-state error due to a ramp input for your choice of $D_i(s)$ in part (a).

(c) if β=0.9, what is the system type for part (b)? What are the values of the appropriate error constant?

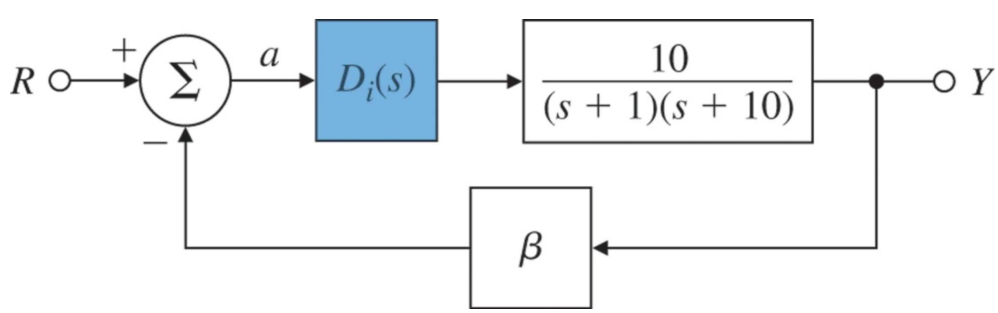


Fig. 4.35.

Solution:

(a) We need at least an integrator in the loop if we desire a Type 1 system. Since there isn't one already, choose $D_2(s)$. The closed-loop transfer function is:

$$T(s) = \frac{Y}{R} = \frac{D_2 G}{1 + D_2 G\beta}$$

Error is:

$$E(s) = R(1-T) = \frac{1 + D_2 G\beta - D_2 G}{1 + D_2 G\beta}$$

At steady state, tracking a ramp signal (aiming for being Type 1), the SS error is:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left(\frac{1}{s^2}\right)\left(\frac{1 + \frac{k_p s + k_I}{s}\frac{10}{(s+1)(s+10)}\beta - \frac{k_p s + k_I}{s}\frac{10}{(s+1)(s+10)}}{1 + \frac{k_p s + k_I}{s}\frac{10}{(s+1)(s+10)}\beta}\right)$$

$$= \lim_{s\to 0} s\left(\frac{1}{s^2}\right)\frac{s(s+1)(s+10) + 10\beta(k_p s + k_I) - 10(k_p s + k_I)}{s(s+1)(s+10) + 10\beta(k_p s + k_I)} \qquad \text{(a)}$$

With $\beta = 1$, the above expression reduces to $e_{ss} = \frac{1}{k_I} \le \frac{1}{10} \Rightarrow k_I \ge 10$.

The other condition that may constrain the choice of gain is stability. The system characteristics equation:

$$A(s) = 1 + D_2 G\beta = 0.$$

It is equivalent to $A(s) = s(s+1)(s+10) + 10\beta(k_p s + k_I) = s^3 + 11s^2 + 10(1+\beta k_p)s + 10\beta k_I = 0$

Use Routh stability method and with $\beta = 1$

$$\begin{array}{lll} s^3 & 1 & 10(1+k_p) \\ s^2 & 11 & 10k_I \\ s^1 & \frac{110(1+k_p) - 10k_I}{11} & \\ s^0 & 10k_I & \end{array}$$

which requires $k_I > 0$ and $k_p > \frac{k_I}{11} - 1$ Combining with the previous condition on steady state error,

$$k_I > 10 \text{ and } k_p > \frac{k_I}{11} - 1$$

(b) $\beta = 0.9$ and $R = \frac{1}{s^2}$, Equation (a) becomes

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)\frac{s(s+1)(s+10) + 10\beta(k_p s + k_I) - 10(k_p s + k_I)}{s(s+1)(s+10) + 10\beta(k_p s + k_I)} = \infty$$

The system is no longer Type 1.

Note: when $\beta = 0.9$, The system does not have a unity feedback. The number of integrators in the system does not necessarily equal to the number of system Type.

(c) Start with $R = \frac{1}{s}$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left(\frac{1}{s}\right)\frac{s(s+1)(s+10) + 10(0.9)(k_p s + k_I) - 10(k_p s + k_I)}{s(s+1)(s+10) + 10(0.9)(k_p s + k_I)} = \frac{-k_I}{9k_I} = -\frac{1}{9} = -\frac{1}{1+8}$$

So the system is Type 0, with a position constant $K_p = 8$.

Note: Ignore the minus sign. A negative steady-state error means the steady-state response is higher than the reference level. It is the absolute values of the error we are aiming to control.

7. The DC motor speed control shown in Fig. 4.41 is described by the differential equation

$$\dot{y} + 60y = 600v_a - 1500w$$

Where y is the motor speed, v_a is the armature voltage, and w is the load torque. Assume that the armature voltage is computed by using the PI control law

$$v_a = k_p e + k_I \int_0^t e dt \, .$$

where $e = r - y$

(a) Compute the transfer function from W to Y as a function of k_p and k_I.

(b) Compute values for k_p and k_I so that the characteristic equation of the closed-loop system will have roots at $-60 \pm 60j$

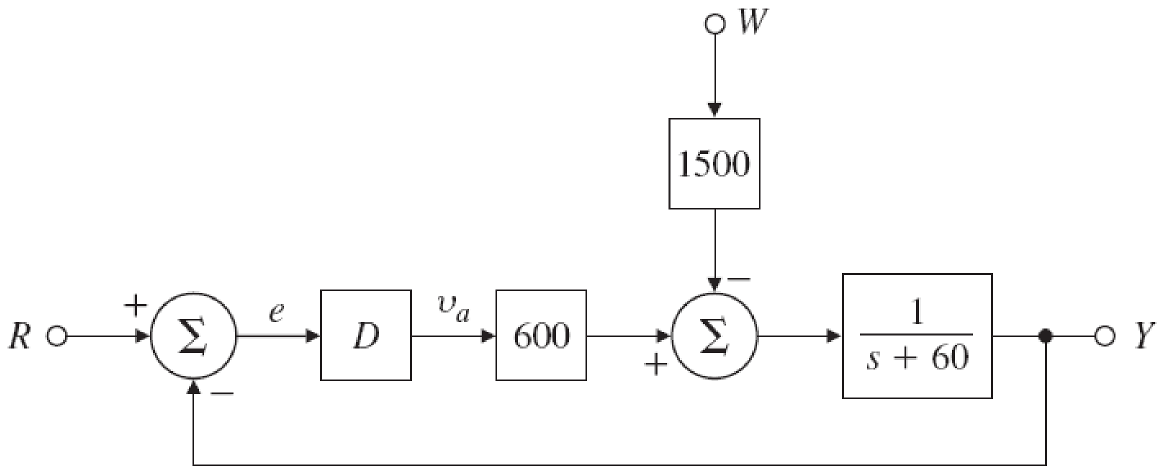


Fig. 4.41

Solution:

(a) Transfer function $T_w = \frac{Y}{W}$, denote $G(s) = \frac{1}{s+60}$; $D(s) = k_p + \frac{k_I}{s}$; set $R=0$ when deriving $T_w = \frac{Y}{W}$:

$$Y = G(s)(600V_a - 1500W) = G(s)(600DE - 1500W) = G(s)(600D(R-Y) - 1500W) = -600GDY - 1500GW$$

$$Y = \frac{-1500G(s)W}{1+600G(s)D(s)}; \qquad \frac{Y}{W} = \frac{-1500\frac{1}{s+60}}{1+600\left(\frac{1}{s+60}\right)\left(k_p + \frac{k_I}{s}\right)} = \frac{-1500s}{s^2 + 60(1+10k_p)s + 600k_I} \quad (1)$$

b) For roots at $-60 \pm 60j$, the denominator of the above transfer function must give terms

$$(s+60-60j)(s+60+60j) = (s+60)^2 + 60^2 = s^2 + 120s + 2 \cdot 60^2 \quad (2)$$

Compare (2) and (1) and equate like power terms:

$600k_I = 2 \cdot 60^2$; $60(1+10k_p) = 120 \Rightarrow k_I = 12; k_p = 0.1$