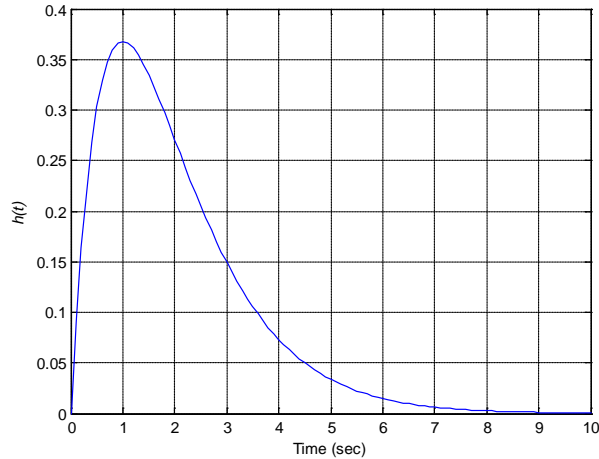


Assignment #2

1. The impulse response of a system is given analytically by $h(t)$ and shown in the figure:

$$h(t) = \begin{cases} te^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



- What is the system's transfer function?
- What is the step response $Y(s)$?
- Through the inverse Laplace transform of $Y(s)$, find the time domain step response function $y(t)$.

Solution:

a) The transfer function of the system is the Laplace transform of the impulse response

$$H(s) = L[te^{-t}] = -\frac{d}{ds} L[e^{-t}] = -\frac{d}{ds} \left(\frac{1}{s+1} \right) = \frac{1}{(s+1)^2}$$

b) Therefore, the step response is $Y(s) = H(s)U(s)$ where $U(s) = \frac{1}{s}$

$$Y(s) = \frac{1}{s(s+1)^2}$$

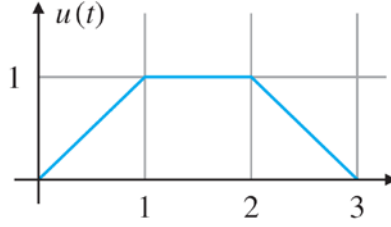
$$c) y(t) = L^{-1} \left[\frac{1}{s(s+1)^2} \right] = L^{-1} \left[\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1} \right] = (1 - (t+1)e^{-t}) \cdot 1(t)$$

$$\therefore y(t) = (1 - (t+1)e^{-t}) \cdot 1(t)$$

2. Consider the standard second-order system (assuming under-damped)

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Write the Laplace transform of the signal in figure below.
- What is the Laplace transform of the output if this signal is applied to $G(s)$?
- Find the output of the system for the input shown in the figure.



Solution:

(a) Let $r(t) = t \cdot 1(t)$ be a ramp signal applied at $t \geq 0$, then

$$u(t) = r(t) - r(t-1) - r(t-2) + r(t-3)$$

The Laplace transform of the above signal is: $U(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$

(b) First find the Laplace transform of the output in response to a ramp signal applied at the start of time $t=0$:

$$Y_1(s) = G(s) \frac{1}{s^2} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \frac{1}{s^2} = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3 s + C_4}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where

$$C_1 = [s^2 Y_1(s)]|_{s=0} = 1$$

Recovering the numerator of ω_n^2 suggests the following relations:

$$C_1(s^2 + 2\xi\omega_n s + \omega_n^2) + C_2 s(s^2 + 2\xi\omega_n s + \omega_n^2) + (C_3 s + C_4)s^2 = \omega_n^2$$

Therefore,

$$C_2 = -\frac{2\xi}{\omega_n}; C_3 = -C_2 = \frac{2\xi}{\omega_n}; C_4 = 4\xi^2 - 1$$

And

$$\begin{aligned} Y_1(s) &= \frac{1}{s^2} - \frac{2\xi/\omega_n}{s} + \frac{2\xi/\omega_n s + 4\xi^2 - 1}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{s^2} - \frac{2\xi/\omega_n}{s} + \frac{2\xi/\omega_n (s + \xi\omega_n) + 2\xi^2 - 1}{(s + \xi\omega_n)^2 + (\omega_n \sqrt{1 - \xi^2})^2} \\ &= \frac{1}{s^2} - \frac{2\xi/\omega_n}{s} + \frac{2\xi/\omega_n (s + \xi\omega_n)}{(s + \xi\omega_n)^2 + (\omega_n \sqrt{1 - \xi^2})^2} + \frac{2\xi^2 - 1}{\omega_n \sqrt{1 - \xi^2}} \frac{\omega_n \sqrt{1 - \xi^2}}{(s + \xi\omega_n)^2 + (\omega_n \sqrt{1 - \xi^2})^2} \end{aligned}$$

The output to input $U(s)$ is thus:

$$Y(s) = Y_1(s) - Y_1(s)e^{-s} - Y_1(s)e^{-2s} + Y_1(s)e^{-3s}$$

(c) The time function of the response is

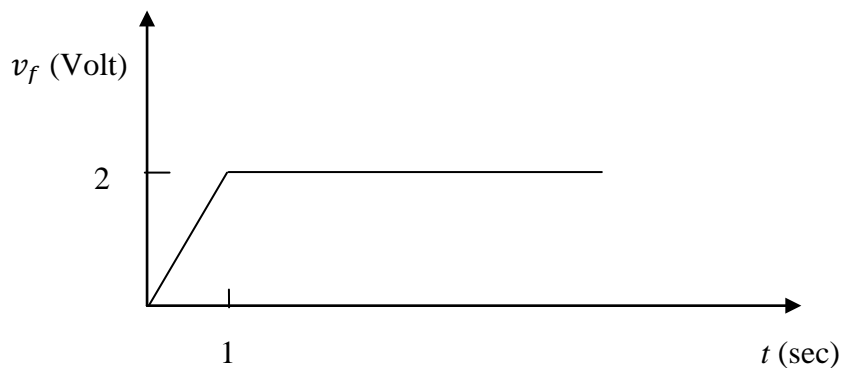
$$y_1(t) = \left[t - \frac{2\xi}{\omega_n} + \frac{2\xi}{\omega_n} e^{-\xi\omega_n t} \cos \omega_d t + \frac{2\xi^2 - 1}{\omega_d} e^{-\xi\omega_n t} \sin \omega_d t \right] \cdot 1(t), \text{ where } \omega_d = \omega_n \sqrt{1 - \xi^2}$$

$$y(t) = y_1(t) - y_1(t-1) - y_1(t-2) + y_1(t-3)$$

3. A rotating load is connected to a field-controlled DC motor with negligible field inductance. The output load speed ω and input voltage v_f follow the relation: $\tau\dot{\omega} + \omega = Kv_f$, where τ and K are constants. An impulse test, (that is, $v_f = \delta(t)$), shows the response is:

$$\omega(t) = \begin{cases} 100e^{-10t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- What are the values of τ and K ?
- What is the steady state load speed to a constant voltage input of 2 volts applied at $t=0$?
- Find the response function of the load speed to a unit ramp input voltage. Do not substitute values of τ and K in your transfer function.
- Write down the input function as depicted in the figure below.
- Use your findings from c) and d), find the load speed value at $t=3$.



a) Transfer function from the given ODE is: $\frac{\Omega}{V_f} = \frac{K}{\tau s + 1} \stackrel{\text{def}}{=} H(s)$;

On the other hand, the Laplace transform of an impulse response is also the transfer function: $H(s) = \mathcal{L}[\omega(t)] = \mathcal{L}[100e^{-10t}] = \frac{100}{s+10}$; Therefore,

$$K = 10; \tau = 0.1$$

b) Use the final value theorem:

$$\omega_{ss} = \lim_{s \rightarrow 0} s \left(\frac{2}{s} \right) \frac{100}{s+10} = 20 \text{ rad/s}$$

c) The response to a unit ramp input is

$$\Omega_1(s) = H(s) \frac{1}{s^2} = \frac{K}{(\tau s + 1)s^2} = \frac{K}{s^2} - \frac{K\tau}{s} + \frac{K\tau^2}{\tau s + 1}$$

$$\omega_1(t) = \mathcal{L}^{-1}[\Omega_1(s)] = K \left[t - \tau + \tau e^{-\frac{t}{\tau}} \right] \cdot 1(t)$$

d) The depicted input function is:

$$v_f = 2t \cdot 1(t) - 2(t-1) \cdot 1(t-1)$$

e) According to LTI system's property, the output is

$$\omega(t) = 2\omega_1(t) - 2\omega_1(t-1)$$

Therefore, at $t=3$, the load speed is

$$\omega(3) = 2\omega_1(3) - 2\omega_1(2)$$

$$\omega_1(3) = 10(3 - 0.1 + 0.1e^{-30}) = 29 \text{ rad/s}$$

$$\omega_1(2) = 10(2 - 0.1 + 0.1e^{-20}) = 19 \text{ rad/s}$$

$$\Rightarrow \omega(3) = 20 \text{ rad/s}$$

4. A rotating load is connected to a field-controlled DC motor with negligible field inductance. A test results in the output load reaching a speed of 1 rad/sec within 1/2 sec when a constant input of 100 V is applied to the motor terminals. The output steady-state speed from the same test is found to be 2 rad/sec. Determine the transfer function $\theta(s)/V_f(s)$ of the motor.

Solution:

A DC motor speed control with negligible field inductance can be described by a first order system: that is: $\Omega(s) = \frac{K}{\tau s + 1} V_f(s)$ where $\Omega(s)$ and $V_f(s)$ are the Laplace transform of motor speed and input voltage, respectively.

For a constant input with magnitude A (A=100 here), $V_f(s) = \frac{A}{s}$;

The steady-state speed is:

$$\omega_{ss} = \lim_{s \rightarrow 0} s \Omega(s) = \lim_{s \rightarrow 0} s \frac{K}{\tau s + 1} V_f(s) = \lim_{s \rightarrow 0} s \frac{K}{\tau s + 1} \frac{A}{s} = KA = 2 \quad \Rightarrow KA = 2 \Rightarrow K = 0.02$$

The time domain response is

$$\omega(t) = L^{-1}[\Omega(s)] = L^{-1}\left[\frac{K}{\tau s + 1} \frac{A}{s}\right] = L^{-1}\left[KA \left(\frac{1}{s} - \frac{1}{s - \frac{1}{\tau}}\right)\right] = KA \left(1(t) - e^{-\frac{t}{\tau}}\right)$$

Given also the speed at 0.5 sec, we have

$$\omega(0.5) = KA \left(1(0.5) - e^{-\frac{0.5}{\tau}}\right) = 2 \left(1 - e^{-\frac{0.5}{\tau}}\right) = 1 \quad \Rightarrow \tau = -\frac{1}{2 \ln 0.5} = 0.7213$$

Therefore, the transfer function of motor “position” in response to voltage is:

$$\frac{\theta(s)}{V_f(s)} = \frac{\Omega(s)/s}{V_f(s)} = \frac{K}{(\tau s + 1)s} = \frac{0.02}{(0.7213s + 1)s}$$

5. For a second-order system with transfer function

$$G(s) = \frac{3}{s^2 + 2s - 3}$$

Determine the following:

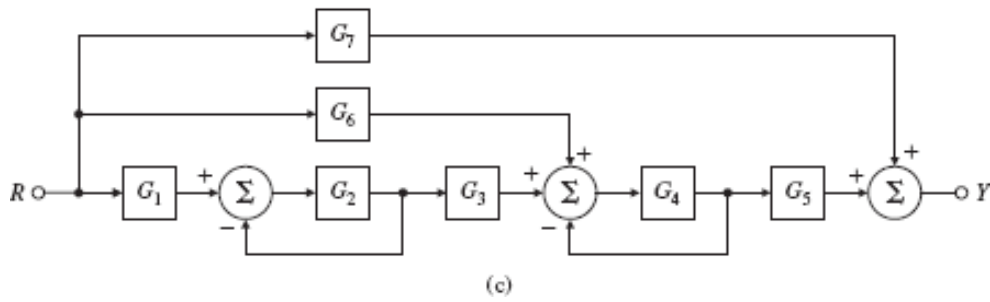
- (a) The DC gain;
- (b) The final value to a step input.

Solution:

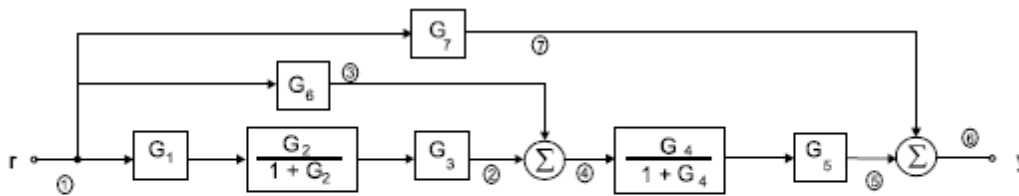
- (a) The DC gain is $G(0) = -1$

(b) Since the transfer function has poles located at $p_1 = -3, p_2 = 1$, there is a pole located in the right half plane. The system is unstable and a step response is not bounded. Final value theorem does not apply in such case. Unlike stable system, DC gain for unstable system is not useful in determining the steady state (final state). The final value of a step input for an unstable system is unbounded (infinity).

6. Find the transfer function for the block diagram:

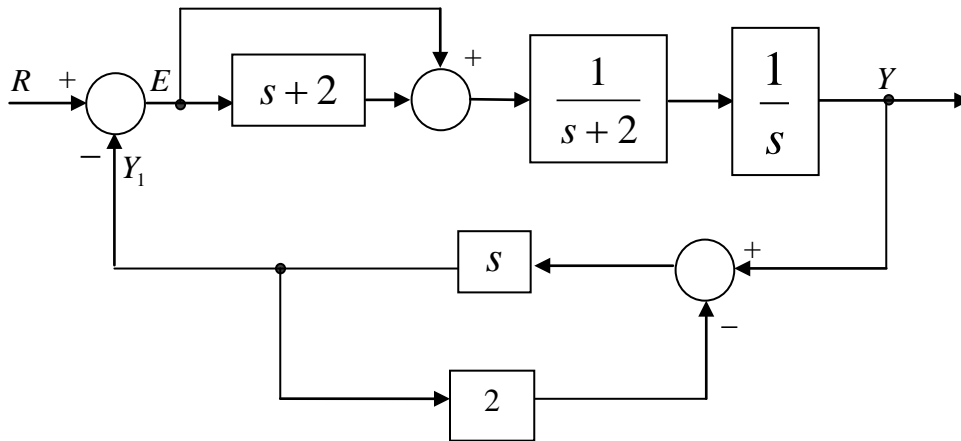


Solution:



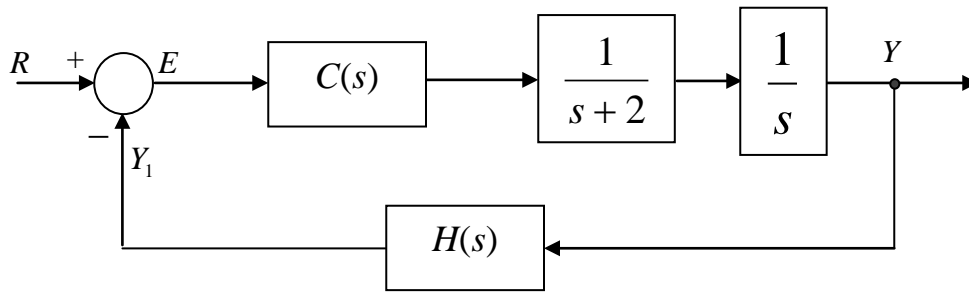
$$\frac{Y}{R} = G_7 + \frac{G_6 G_4 G_5}{1 + G_4} + \frac{G_1 G_2 G_3}{1 + G_2} \times \frac{G_4 G_5}{1 + G_4}$$

7. Find the transfer function of $\frac{Y(s)}{R(s)}$



Solution:

The above block diagram is equivalent to the following:



where

$$C(s) = s + 2 + 1 = s + 3$$

$$H(s) = \frac{s}{1 + 2s}$$

Hence,

$$\frac{Y(s)}{R(s)} = \frac{C(s) \frac{1}{s+2} \cdot \frac{1}{s}}{1 + C(s)H(s) \frac{1}{s+2} \cdot \frac{1}{s}}$$

8. Suppose you are to design a unity feedback controller for a first-order plant depicted in Fig. 3.59. As you will learn in Chapter 4, the configuration shown is referred to as a proportional-integral controller. You are to design the controller so that the closed-loop poles lie within the shaded regions shown in Fig. 3.60.

- (a) What values of ω_n and ξ correspond to the shaded regions in Fig. 3.60 (A simple estimate from the figure is sufficient).
- (b) Let $K_\alpha = \alpha = 2$. Find values for K and K_I so that the poles of the closed-loop system lie with the shaded regions.

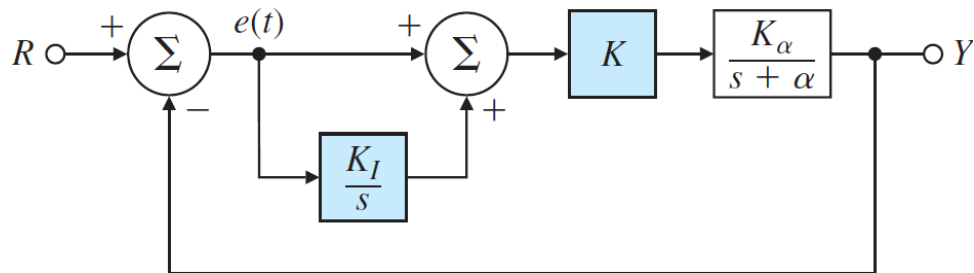


Fig. 3.59

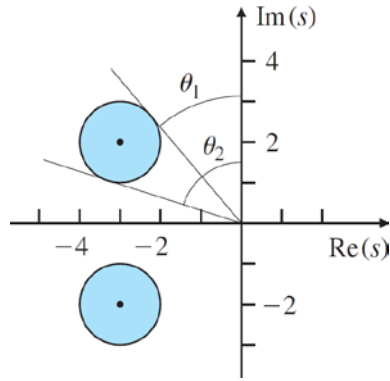


Fig. 3.60

Solution:

a) The center of the region appears to be at $\sqrt{3^2 + 2^2} = 3.6$, we can roughly say that $2.6 \leq \omega_n \leq 3.6$. We can also read the angles to determine the range for damping ratio:

$$\theta_1 = \tan^{-1}\left(\frac{2}{2.5}\right) = 38.6^\circ; \quad \xi_1 = \sin 38.6^\circ = 0.62$$

$$\theta_2 = \tan^{-1}\left(\frac{3}{1}\right) = 71.6^\circ; \quad \xi_2 = \sin 71.6^\circ = 0.95 \quad \text{Therefore: } 0.62 \leq \xi \leq 0.95$$

b) Closed-loop pole locations are defined as the roots of: $1 + \left(1 + \frac{K_I}{s}\right)K\left(\frac{K_\alpha}{s + \alpha}\right) = 0$ which is the denominator of the closed-loop TF;

$$s^2 + (\alpha + KK_\alpha)s + KK_I K_\alpha = 0$$

With $K_\alpha = \alpha = 2$ given in this question, the above reduces to: $s^2 + 2(1 + K)s + 2KK_I = 0$

If we choose the center of the shaded region to be the locations of poles, the poles are the roots of

$$(s + 3)^2 + 2^2 = 0 \quad \text{or equivalently, } s^2 + 6s + 13 = 0$$

Consequently, $1 + K = 3; 2KK_I = 13$

Therefore, $K = 2; K_I = \frac{13}{4}$

9. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(s + 2)}$$

The desired system response to a step input is specified as peak time $t_p = 1$ sec and overshoot is less than 5%.

- (a) Determine whether both specifications can be met simultaneously by selecting the right value of K .
- (b) Sketch the associated region in the s-plane where both specifications are met, and indicate what root locations are possible for some likely values of K .

Solution:

Peak time $t_p = 1$ determines the damped natural frequency: $\omega_d = \frac{\pi}{t_p} = \pi$, where $\omega_d = \omega_n \sqrt{1 - \xi^2}$.

Having a peak time implies that the system is underdamped with overshoot.

Overshoot $M_p = 5\%$ determines the damping coefficient: $\xi = \frac{[\ln M_p]^2}{\pi^2 + [\ln M_p]^2} = 0.69$.

(a) The closed-loop transfer function with a unity feedback is:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\frac{K}{s(s+2)}}{1 + \frac{K}{s(s+2)}} = \frac{K}{s^2 + 2s + K}$$

Match with a standard second order system: $T(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$, equate the coefficients of like power terms of s's:

$$\begin{aligned} \frac{K}{2} = \xi\omega_n & \Rightarrow \omega_n = \sqrt{K} \\ \xi = \frac{1}{\sqrt{K}} & \Rightarrow \omega_d = \omega_n \sqrt{1 - \xi^2} = \sqrt{K - 1} \end{aligned}$$

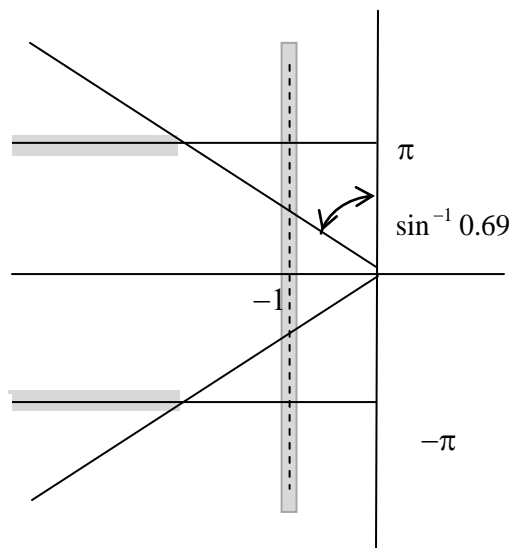
To achieve peak time of 1 sec, $\omega_d = \sqrt{K - 1} = \pi \Rightarrow K = \pi^2 + 1$; However with this value of K :

$\xi = \frac{1}{\sqrt{K}} = \frac{1}{\sqrt{\pi^2 + 1}} = 0.3 < 0.69$; The overshoot requirement is not met. Therefore, we cannot meet both requirements by choosing a single value of K .

(b). For both requirements to be met, the poles have to lie on the horizontal line of $\pm \pi j$ and inside the cone of 0.69 damping. To see what solutions are possible, we work out the pole location in terms of K :

$$p_{1,2} = -1 \pm j\omega_d = -1 \pm j\sqrt{K - 1}$$

Apparently, with different values of K , closed-loop poles maintain the same real part while only the imaginary coordinates change. The vertical line at -1 indicates the possible pole locations. It is clear that if one wants to satisfy damping requirement, one has to allow much longer peak time than 1 second. Conversely, if one wants to put more restriction on peak time, one has to allow much higher overshoot than 5%. It becomes a trade-off in this case unless a different control is considered.



10. The equations of motion for the DC motor shown in Fig. 2.22 were given in Eqs. (2.45-2.46) as

$$J_m \ddot{\theta}_m + (b + \frac{K_t K_e}{R_a}) \dot{\theta}_m = \frac{K_t}{R_a} v_a$$

Assume that

$$J_m = 0.01 \text{ kg} \cdot \text{m}^2 ; b = 0.001 \text{ N} \cdot \text{m} \cdot \text{sec} ; K_e = 0.02 \text{ V} \cdot \text{sec} ; K_t = 0.02 \text{ N} \cdot \text{m/A} ; R_a = 10 \Omega$$

- (a) Find the transfer function between the applied voltage v_a and motor speed $\dot{\theta}_m$.
- (b) What is the steady-state speed of the motor after a voltage $v_a = 10 \text{ V}$ has been applied?
- (c) Find the transfer function between the applied voltage v_a and the shaft angle θ_m .
- (d) Suppose feedback is added to the system in part (c) so that it becomes a position servo device such that the applied voltage is given by

$$v_a = K(\theta_r - \theta_m).$$

where K is the feedback gain. Find the transfer function between θ_r and θ_m .

- (e) What is the maximum value of K that can be used if an overshoot $M_p < 20\%$ is desired?
- (f) What values of K will provide a rise time of less than 4 sec? (Ignore the M_p constraint).

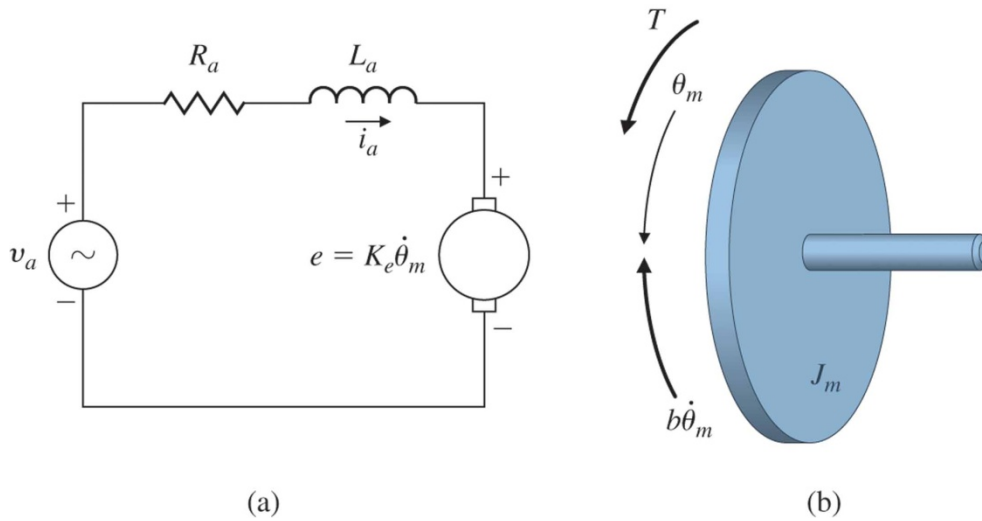


Fig. 2.22

Solution:

- (a) Let $\omega_m = \dot{\theta}_m$, since $\ddot{\theta}_m = \frac{d}{dt} \dot{\theta}_m$, the governing equation of the motor can be written as

$$J_m \dot{\omega}_m + (b + \frac{K_t K_e}{R_a}) \omega_m = \frac{K_t}{R_a} v_a$$

The transfer function between the applied voltage and motor speed is

$$\frac{\Omega_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_a J_m}}{s + \frac{b}{J_m} + \frac{K_t K_e}{R_a J_m}}$$

Using the given values of motor constants, the transfer function is

$$\frac{\Omega_m(s)}{V_a(s)} = \frac{0.2}{s + 0.104}$$

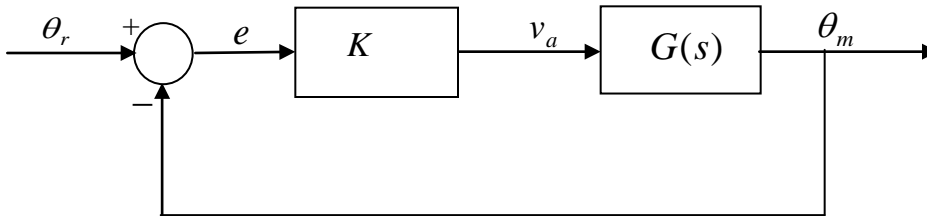
(b) The transfer function indicates a stable system. Using the final value theorem

$$\omega_m(\infty) = \lim_{s \rightarrow 0} s \Omega_m(s) = \lim_{s \rightarrow 0} s V_a(s) \frac{0.2}{s + 0.104} = \lim_{s \rightarrow 0} s \left(\frac{10}{s} \right) \frac{0.2}{s + 0.104} = 19.23 \text{ rad/s}$$

(c) The transfer function between the applied voltage and motor position is

$$G(s) = \frac{\Xi_m(s)}{V_a(s)} = \frac{\Omega_m(s)}{s V_a(s)} = \frac{0.2}{s(s + 0.104)}$$

(d) The best way to finding the closed-loop transfer function is through the aid of a block diagram



$$T(s) = \frac{\Xi_m(s)}{\Xi_r(s)} = \frac{KG(s)}{1 + KG(s)} = \frac{0.2K}{s^2 + 0.104s + 0.2K}$$

(e) $M_p < 20\%$ translate to $\xi > 0.4559$

From the closed-loop transfer function $T(s)$, we have

$$\frac{\omega_n^2}{2\xi\omega_n} = 0.104 \Rightarrow \xi = \frac{0.104}{2\sqrt{0.2K}} > 0.4559 \Rightarrow K < 0.065$$

(f)

$$\omega_n = 0.2K \geq \frac{1.8}{t_r}$$

$$K \geq 1.01$$

11. Consider the system shown in Fig. 3.64, where

$$G(s) = \frac{1}{s(s + 3)} \quad \text{and} \quad D(s) = \frac{K(s + z)}{s + p}$$

Find K , z , and p so that the closed-loop system has a 10% overshoot to a step input and a settling time of 1.5 sec (1% criterion).

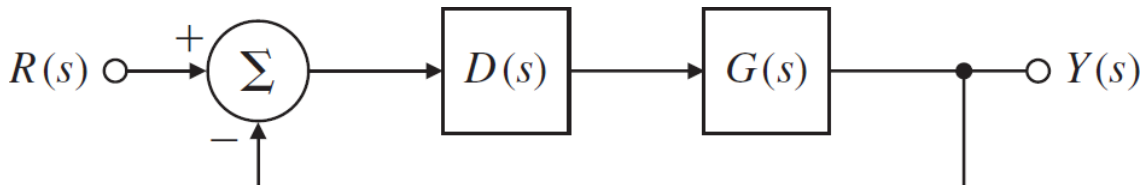


Fig. 3.64

Solution:

With 10% overshoot:

$$\zeta = \frac{|\ln M_p|}{\sqrt{\pi^2 + (\ln M_p)^2}} = 0.6$$

For the 1.5 sec (1% criterion):

$$\sigma = \frac{4.6}{t_s} = 3.07$$

The closed-loop transfer function is

$$T(s) = \frac{D(s)G(s)}{1 + D(s)G(s)} = \frac{\frac{K(s+z)}{s+p} \frac{1}{s(s+3)}}{1 + \frac{K(s+z)}{s+p} \frac{1}{s(s+3)}} = \frac{K(s+z)}{s(s+3)(s+p) + K(s+z)} = \frac{K(s+z)}{s^3 + (3+p)s^2 + (3p+K)s + Kz} \quad (1)$$

Obviously, this is a 3rd order TF, there are only two possibilities: a) all three poles are real and b) one real and a pair of complex poles. Since the design specification indicates a certain percentage of overshoot, we can safely aim at option b). In such case, the denominator of $T(s)$ (let it be $A(s)$) can be written in the form:

$$A(s) = (s+a)(s^2 + 2\sigma s + \omega_n^2) = s^3 + (2\sigma + a)s^2 + (\omega_n^2 + 2\sigma a)s + a\omega_n^2 \quad (2)$$

It is convenient to put the real pole far away from the complex poles, so that we can just deal with the dominant poles. Knowing that $\sigma = 3.07$, we can put the real pole at -15, that is, $a=15$

It is better to re-write ω_n as $\omega_n = \frac{\sigma}{\zeta}$, use $\zeta = 0.7$, $a=15$, and $\sigma = 3.1$ to be safe, (2) becomes

$$A(s) = s^3 + 21.2s^2 + 112.6s + 294.2$$

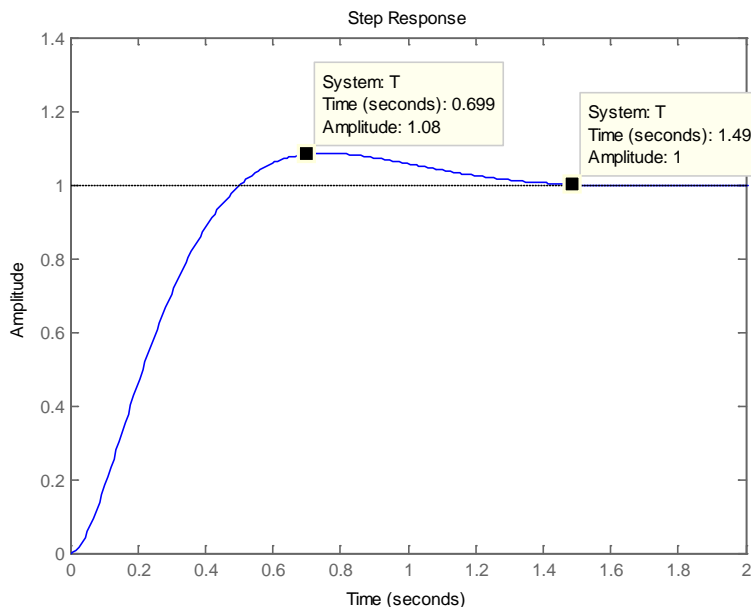
Equate the like powers of s with the denominator in eq. (1):

$$p = 18.2$$

$$K = 52$$

$$z = 5.1$$

Using the above parameters in the transfer function (1), we can simulate its step response in matlab, the response is plotted below. The overshoot is 8%, and settling in 1% region at $t=1.48$. Design goals are met.



12. Sketch the step response of a system with the transfer function

$$G(s) = \frac{s/2 + 1}{(s/40 + 1)[(s/4)^2 + s/4 + 1]}$$

Justify your answer on the basis of the locations of the poles and zeros. (Do not find the inverse Laplace transform). Then compare your answer with the step response computed using Matlab.

Solution:

The system has three poles located at $p_1 = -40$, $p_{2,3} = -2 \pm j\sqrt{12}$ and one zero at: -2.

Since the real pole at -40 is much faster than the complex poles (real parts are more than factor of 4 apart), it can be ignored. The original TF can be approximated by removing the real pole:

$$G_2(s) = \frac{s/2 + 1}{[(s/4)^2 + s/4 + 1]}$$

One can see from the simulated response how close the responses are (blue and green curves).

To sketch the response by hand (without the aid of Matlab simulation), we start with the complex poles only: (when doing so, make sure that the same DC-gain is maintained)

$$G_3(s) = \frac{1}{[(s/4)^2 + s/4 + 1]} = \frac{4^2}{s^2 + 4s + 4^2}$$

Now since the real part of the zero is close to that of the complex poles, zero will increase the response overshoot but not affecting the settling time too much. Compare the red and green curves to observe the effects.

