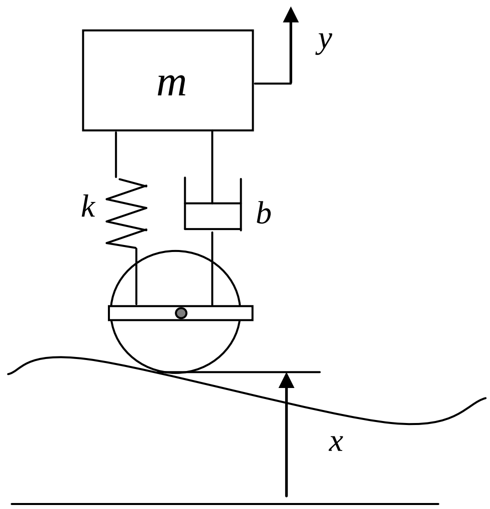
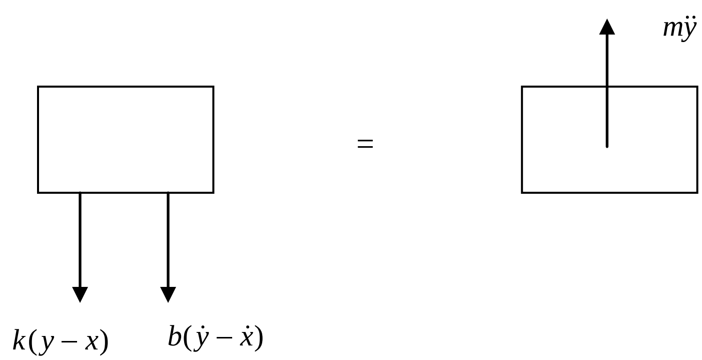
ENME 585 Assignment #1 Solution

1. A quarter car suspension system is shown in the figure below. In the figure, *y* is the vertical displacement measured from the car static equilibrium in a fixed reference frame (inertia frame); *x* is the road height. Model parameters are mass *m* [kg], spring constant *k* [N/m], and damping coefficient *b* [N s/m]. Derive the differential equation that relates *y* to *x*.



Solution:

FBD



Using Newton's Second Law of Motion:

$$-k(y-x)-b(\dot{y}-\dot{x})=m\ddot{y}$$

Re-arranging:

$$\boxed{m\ddot{y}+b\dot{y}+ky=b\dot{x}+kx}$$

2. Continuing from Question 1, we now consider the idea of active suspension by adding a hydraulic actuator situated in parallel to the damper. The actuator generates a force *u* to the bottom of the car body. Assuming also that this force follows such a control law: $u=-k_p y$ where *y* is the vertical displacement relative to some fixed reference and $k_p > 0$ is the controller gain.

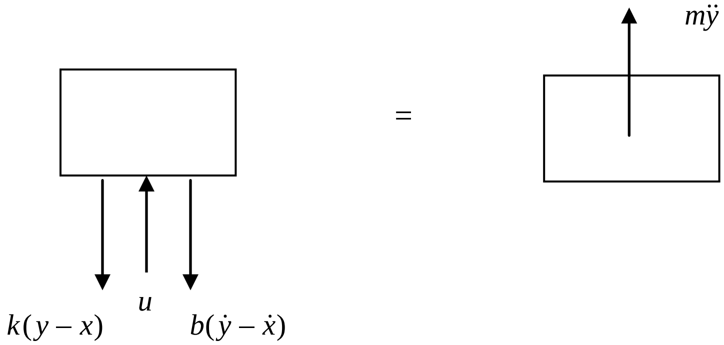
1) What is the desirable output (control goal)?
2) What kind of sensor can we use to measure the output?
3) Speculate whether this control law would improve the suspension quality (use your mechanical engineer's intuition);
4) If your answer to 3) is negative, how would you improve by modifying the control law?
5) Draw a block diagram to show the feedback control system.

Solution:

1) The desirable output or control goal is $y=0$. In other words, regardless how bumpy the road surface is, we want the car body not to move vertically with respect to a fixed observer standing by the road;

2) A position sensor like a laser device set up by the road side can measure the absolute height of the car, which means the car cannot carry the sensor and it has to communicate with the laser device. This is not convenient and may be expensive.

3) FBD



Again using Newton's second law of motion, the 1/4 car model with u applied to the car body follows this equation:

$$m\ddot{y}+b\dot{y}+ky=b\dot{x}+kx+u$$

With $u=-k_p y$ the closed loop system follows:

$$m\ddot{y}+b\dot{y}+(k+k_p)y=b\dot{x}+kx$$

The net effect is that the car is made stiffer by the controller because $k_p>0$ and therefore the spring constant is increased. This may not be what we want. A softer spring is more desirable because it means better riding comfort and better suspension. So we speculate that this control law would not improve the suspension quality.

4) Use our mechanical engineers' intuition, if we increase the dynamic mass or inertia of the car, the car will be more robust to road disturbance. We try the following control law:

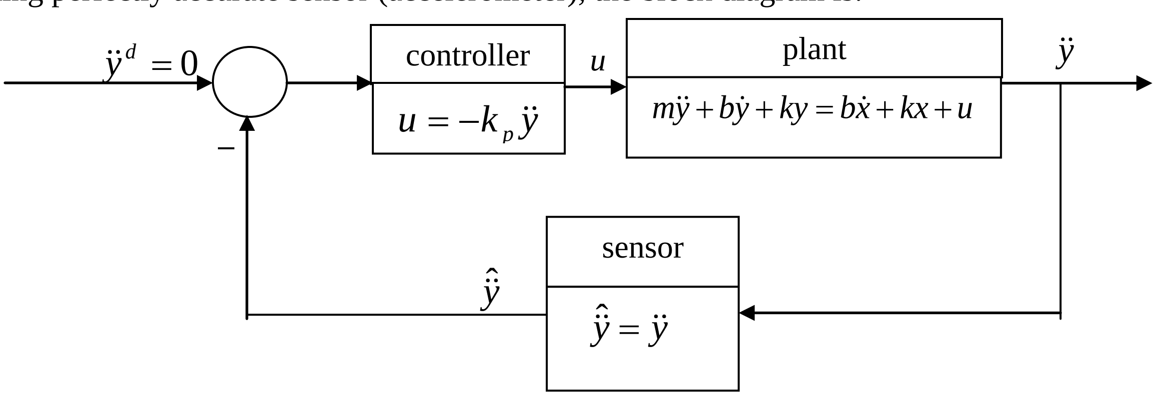
$$u=-k_p\ddot{y}$$

which implies that we need to measure vertical acceleration of the car. This can be conveniently done via an on-board accelerometer. With this law, the closed-loop system follows:

$$(m+k_p)\ddot{y}+b\dot{y}+ky=b\dot{x}+kx$$

It seems that we can make the equivalent mass anything we want. As if the car is really heavy, it will not react to road disturbance, meaning better suspension.

5) Assuming perfectly accurate sensor (accelerometer), the block diagram is:



3. Prove the following ODE describes a linear system:

$$\ddot{y} + y = 2u$$

where u is input and y is output.

Solution:

Check condition #1: superposition

$u_1 \rightarrow y_1: \ \ddot{y}_1 + y_1 = 2u_1 \quad (1)$

$u_2 \rightarrow y_2: \ \ddot{y}_2 + y_2 = 2u_2 \quad (2)$

Summing (1) and (2), the following must be true:

$\ddot{y}_1 + \ddot{y}_2 + y_1 + y_2 = 2(u_1 + u_2) \quad (3)$

$u_\Sigma = u_1 + u_2 \rightarrow y_\Sigma: \ \ddot{y}_\Sigma + y_\Sigma = 2u_\Sigma = 2(u_1 + u_2) \quad (4)$

Comparing the left hand side of eq. (3) and (4), it indicates: $y_\Sigma = y_1 + y_2$

Therefore, superposition applies;

Now check condition #2: homogeneity

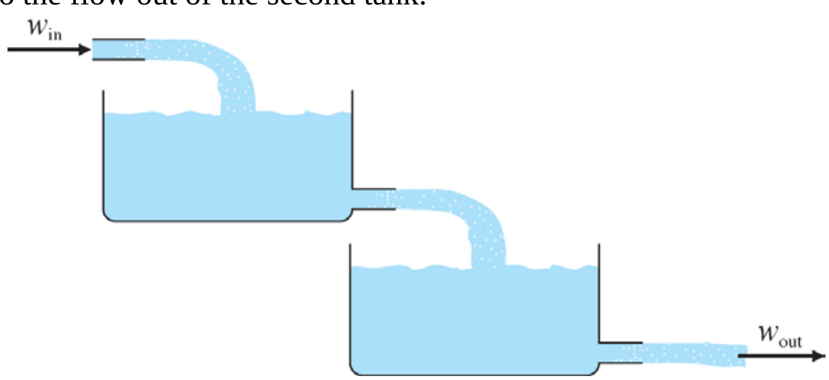
$u_1 \rightarrow y_1: \ \ddot{y}_1 + y_1 = 2u_1 \quad (1)$; Multiplying both sides by α gives: $\alpha\ddot{y}_1 + \alpha y_1 = 2\alpha u_1 \quad (5)$

$\alpha u_1 \rightarrow y_\alpha: \ \ddot{y}_\alpha + y_\alpha = 2\alpha u_1 \quad (6)$

Comparing the left hand side of (5) and (6) indicates: $y_\alpha = \alpha y_1$; meaning $\alpha u_1 \rightarrow \alpha y_1$

Therefore, homogeneity applies. Since both conditions are satisfied, the system is linear!

4. For the two-tank fluid flow system shown in the figure below, find the differential equations relating the flow into the first tank to the flow out of the second tank.



Solution:

From the relation between the height of the water and mass flow rate, the mass conservation states:

$$\dot{m}_1 = \rho A_1 \dot{h}_1 = w_{in} - w$$

$$\dot{m}_2 = \rho A_2 \dot{h}_2 = w - w_{out}.$$

Assuming turbulent flow at the orifice,

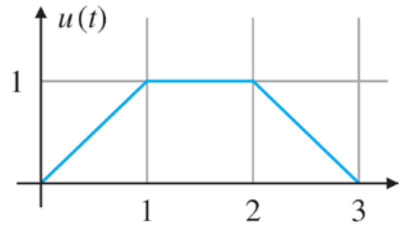
$$w = \frac{1}{R_1}(\rho g h_1)^{\frac{1}{2}}$$

$$w_{out} = \frac{1}{R_2}(\rho g h_2)^{\frac{1}{2}}$$

Finally,

$$\dot{h_1} = -\frac{1}{\rho A_1 R_1}(\rho g h_1)^{\frac{1}{2}} + \frac{1}{\rho A_1} w_{in}$$
$$\dot{h_2} = \frac{1}{\rho A_2 R_1}(\rho g h_1)^{\frac{1}{2}} - \frac{1}{\rho A_2 R_2}(\rho g h_2)^{\frac{1}{2}}.$$

5. For the waveform shown in the figure:



a) Provide the expression for $u(t)$ using standard input functions introduced in class;
b) Provide the Laplace transform $U(s)$ of the expression obtained in a).

Solution:

(a) Let $r(t) = t \cdot 1(t)$ be a ramp signal applied at $t \geq 0$, then

$$u(t) = r(t) - r(t-1) - r(t-2) + r(t-3)$$
$$= t \cdot 1(t) - (t-1) \cdot 1(t-1) - (t-2) \cdot 1(t-2) + (t-3) \cdot 1(t-3)$$

The Laplace transform of the above signal is: $U(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$

6. Find the Laplace transform of the following time functions:

(a) $f(t) = \sin 2t + 2\cos 2t + e^{-t}\sin 2t$

(b) $f(t) = 1(t) + 2t\cos 2t$

(c) $f(t) = \sin^2 t + 3\cos^2 t$

Solution:

(a) $f(t) = \sin 2t + 2\cos 2t + e^{-t}\sin 2t$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\} + \mathcal{L}\{2\cos 2t\} + \mathcal{L}\{e^{-t}\sin 2t\} = \frac{2}{s^2+4} + \frac{2s}{s^2+4} + \frac{2}{(s+1)^2+4}$$

(b) $f(t) = 1(t) + 2t\cos 2t$

$$\mathcal{L}\{1(t)\} = \frac{1}{s}$$
$$\mathcal{L}\{tg(t)\} = -\frac{d}{ds}G(s)$$
$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$
$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{1(t)\} + 2\mathcal{L}\{t\cos 2t\}$$
$$= \frac{1}{s} + 2\left(-\frac{d}{ds}\frac{s}{s^2+4}\right) = \frac{1}{s} - 2\left[\frac{(s^2+4)-(2s)s}{(s^2+4)^2}\right] = \frac{1}{s} - \frac{2(-s^2+4)}{(s^2+4)^2}.$$

(c) $f(t) = \sin^2 t + 3\cos^2 t$

Use the trigonometric formulas,

$$\sin^2 t = \frac{1-\cos 2t}{2}$$
$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$f(t) = \frac{1-\cos 2t}{2} + 3\left(\frac{1+\cos 2t}{2}\right) = 2 + \cos 2t$$
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2\} - \mathcal{L}\{\cos 2t\} = \frac{2}{s} + \frac{s}{s^2+4} = \frac{3s^2+8}{s(s^2+4)}.$$

7. Find the time function corresponding to each of the following Laplace functions using partial-fraction expansions.

(a) $F(s) = \frac{2}{s(s+2)}$

(b) $F(s) = \frac{10}{s(s+1)(s+10)}$

(c) $F(s) = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}$

(d) $F(s) = \frac{2(s+2)}{(s+1)(s^2+4)}$

(e) $F(s) = \frac{e^{-s}}{s^2}$

(f) $F(s) = \frac{1}{s(s+2)^2}$

(g) $F(s) = \frac{2s^2+s+1}{(s-1)(s^2+s+1)}$

Solutions:

(a) $F(s) = \frac{2}{s(s+2)}$

Perform partial fraction expansion,

$$F(s) = \frac{2}{s(s+2)} = \frac{C_1}{s} + \frac{C_2}{s+2}$$
$$C_1 = \left.\frac{2}{s+2}\right|_{s=0} = 1$$
$$C_2 = \left.\frac{2}{s}\right|_{s=-2} = -1$$
$$F(s) = \frac{1}{s} - \frac{1}{s+2}$$
$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$
$$f(t) = \{1 - e^{-2t}\}1(t).$$

(b) $F(s) = \frac{10}{s(s+1)(s+10)}$

Perform partial fraction expansion,

$$F(s) = \frac{10}{s(s+1)(s+10)} = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+10}$$
$$C_1 = \left.\frac{10}{(s+1)(s+10)}\right|_{s=0} = 1$$
$$C_2 = \left.\frac{10}{s(s+10)}\right|_{s=-1} = -\frac{10}{9}$$
$$C_3 = \left.\frac{10}{s(s+1)}\right|_{s=-10} = \frac{1}{9}$$

$$F(s) = \frac{1}{s} - \frac{\frac{10}{9}}{s+1} + \frac{\frac{1}{9}}{s+10}$$
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \{1 - \frac{10}{9}e^{-t} + \frac{1}{9}e^{-10t}\}1(t).$$

(c) $F(s) = \dfrac{3s^2 + 9s + 12}{(s+2)(s^2 + 5s + 11)}$

Perform partial fraction expansion,

$$F(s) = \frac{3s^2 + 9s + 12}{(s+2)(s^2+5s+11)} = \frac{C_1}{s+2} + \frac{C_2 s + C_3}{s^2 + 5s + 11}$$
$$C_1 = \left.\frac{3(s^2+3s+4)}{(s^2+5s+11)}\right|_{s=-2} = \frac{6}{5}.$$

Equate numerators:

$$\frac{\frac{6}{5}}{(s+2)} + \frac{C_2 s + C_3}{(s^2+5s+11)} = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}$$
$$\left(C_2 + \frac{6}{5}\right)s^2 + (6 + C_3 + 2C_2)s + \left(2C_3 + \frac{66}{5}\right) = 3s^2 + 9s + 12.$$

Equate like powers of s to find C_2 and C_3:

$$C_2 + \frac{6}{5} = 3 \Rightarrow C_2 = \frac{9}{5}$$
$$2C_3 + \frac{66}{5} = 12 \Rightarrow C_3 = -\frac{3}{5}$$

$$F(s) = \frac{\frac{6}{5}}{s+2} + \frac{\frac{9}{5}s - \frac{3}{5}}{s^2+5s+11} = \frac{6}{5}\frac{1}{s+2} + \frac{\frac{9}{5}\left(s+\frac{5}{2}\right) - \frac{51}{10}}{\left(s+\frac{5}{2}\right)^2 + \left(\frac{\sqrt{19}}{2}\right)^2} = \frac{6}{5}\frac{1}{s+2} + \frac{9}{5}\frac{\left(s+\frac{5}{2}\right)}{\left(s+\frac{5}{2}\right)^2 + \left(\frac{\sqrt{19}}{2}\right)^2} - \frac{51}{5\sqrt{19}}\frac{\frac{\sqrt{19}}{2}}{\left(s+\frac{5}{2}\right)^2 + \left(\frac{\sqrt{19}}{2}\right)^2}$$

Therefore, using the Laplace transform table:

$$f(t) = \left[\frac{6}{5}e^{-2t} + \frac{9}{5}e^{-\frac{5}{2}t}\cos\left(\frac{\sqrt{19}}{2}\right)t - \frac{51}{5\sqrt{19}}e^{-\frac{5}{2}t}\sin\left(\frac{\sqrt{19}}{2}\right)t\right]1(t)$$

(d) $F(s) = \dfrac{e^{-s}}{s^2}$

$$F_1(s) = \frac{1}{s^2} \Rightarrow f_1(t) = L^{-1}\left(\frac{1}{s^2}\right) = t1(t)$$
$$f(t) = f_1(t-1) = (t-1)\cdot 1(t-1)$$

(e) $F(s) = \frac{2(s+2)}{(s+1)(s^2+4)}$

$$F(s) = \frac{2(s+2)}{(s+1)(s^2+4)} = \frac{C_1}{s+1} + \frac{C_2 s + C_3}{s^2+4}$$
$$C_1 = (s+1)F(s)|_{s=-1} = \frac{2}{5}$$

Equate numerators and like powers of s terms

$$s^2\left(\frac{2}{5} + C_2\right) + s(C_2 + C_3) + \left(\frac{8}{5} + C_3\right) = 2s + 4$$
$$C_2 = -\tfrac{2}{5};\ C_3 = \tfrac{12}{5}$$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \{\frac{2}{5}e^{-t} - \frac{2}{5}\cos 2t + \frac{6}{5}\sin 2t\}1(t)$

(f) $F(s) = \dfrac{1}{s(s+2)^2}$

Perform partial fraction expansion,

$$F(s) = \frac{1}{s(s+2)^2} = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_3}{(s+2)^2}$$
$$C_1 = sF(s)|_{s=0} = 1/4$$
$$C_3 = (s+2)^2F(s)|_{s=-2} = -\frac{1}{2}$$

Equate numerators and like powers of s terms and sub-in finds of C_1 and C_3 from above

$C_2 = -\frac{1}{4}$;

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \left(\frac{1}{4} - \frac{1}{4}e^{-2t} + \frac{1}{2}te^{-2t}\right)1(t).$$

(g) $F(s) = \frac{2s^2+s+1}{(s-1)(s^2+s+1)}$

Partial fraction expansion:

$$F(s) = \frac{2s^2+s+1}{(s-1)(s^2+s+1)} = \frac{C_1}{s-1} + \frac{C_2 s + C_3}{s^2+s+1}$$
$$C_1 = (s-1)F(s)|_{s=1} = \left.\frac{2s^2+s+1}{s^2+s+1}\right|_{s=1} = \frac{4}{3}$$

Equate numerators and match the coefficients of like powers of s:

$$\frac{\frac{4}{3}}{s-1} + \frac{C_2 s + C_3}{s^2+s+1} = \frac{2s^2+s+1}{(s-1)(s^2+s+1)}$$
$$s^2\left(\frac{4}{3} + C_2\right) + s\left(\frac{4}{3} - C_2 + C_3\right) + \left(\frac{4}{3} - C_3\right) = 2s^2 + s + 1$$
$$\frac{4}{3} + C_2 = 2 \Rightarrow C_2 = \frac{2}{3}$$
$$\frac{4}{3} - C_3 = 1 \Rightarrow C_3 = \frac{1}{3}.$$

$$F(s) = \frac{\frac{4}{3}}{s-1} + \frac{\frac{2}{3}s + \frac{1}{3}}{s^2+s+1} = \frac{\frac{4}{3}}{s-1} + \frac{\frac{2}{3}\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{4}{3}e^t + \frac{2}{3}e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t = \frac{2}{3}\left\{e^t + e^{-\frac{t}{2}}2\cos\frac{\sqrt{3}}{2}t\right\}1(t).$$

8. Solve the following ordinary differential equations using Laplace transforms

(a) $\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0;\ \ y(0) = 1, \dot{y}(0) = 2$

(b) $\ddot{y}(t)+3y(t)=\sin t;\ \ y(0)=1, \dot{y}(0)=2$

(c) $\ddot{y}(t)+y(t)=t;\ \ y(0)=1, \dot{y}(0)=-1$

Solution:

(a) $\ddot{y}(t)+\dot{y}(t)+3y(t)=0;\ \ y(0)=1, \dot{y}(0)=2$

$$s^2\mathrm{Y(s)} - sy(0)-\dot{y}(0)+sY(s)-y(0)+3Y(s)=0$$

$$Y(s)-\frac{s+3}{s^2+s+3}=\frac{\left(s+\frac{1}{2}\right)+\frac{5}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{11}{4}}=\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}+\frac{5}{\sqrt{11}}\frac{\frac{\sqrt{11}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}$$

Using the Laplace Table to obtain:

$$\boxed{y(t)=\left(e^{-t/2}\cos\frac{\sqrt{11}}{2}t+\frac{5}{\sqrt{11}}e^{-t/2}\sin\frac{\sqrt{11}}{2}t\right)\cdot 1(t)}$$

(b) $\ddot{y}(t)+3y(t)=\sin t;\ \ y(0)=1, \dot{y}(0)=2$

$$s^2\mathrm{Y(s)} - sy(0)-\dot{y}(0)+3Y(s)=\frac{1}{s^2+1}$$

$$Y(s)=\frac{1+(s+2)(s^2+1)}{(s^2+1)(s^2+3)}=\frac{s^3+2s^2+s+3}{(s^2+1)(s^2+3)} \qquad (1)$$

A partial fraction expansion can take the following form

$$Y(s)=\frac{C_1s+C_2}{(s^2+3)}+\frac{C_3s+C_4}{(s^2+1)} \qquad (2)$$

Equating coefficients of like power terms in (1) and (2) gives

$$\begin{array}{l} C_1+C_3=1 \\ C_2+C_4=2 \\ C_1+3C_3=1 \\ C_2+3C_4=3 \end{array} \Rightarrow \begin{array}{l} C_1=1 \\ C_2=\frac{3}{2} \\ C_3=0 \\ C_4=\frac{1}{2} \end{array}$$

Therefore, $Y(s)=\dfrac{s+\frac{3}{2}}{(s^2+3)}+\dfrac{\frac{1}{2}}{(s^2+1)}=\dfrac{s}{(s^2+3)}+\dfrac{\sqrt{3}}{2}\dfrac{\sqrt{3}}{(s^2+3)}+\dfrac{1}{2}\dfrac{1}{(s^2+1)}$

$$\boxed{y(t)=(\cos\sqrt{3}t+\frac{\sqrt{3}}{2}\sin\sqrt{3}t+\frac{1}{2}\sin t)\cdot 1(t)}$$

(f) $\ddot{y}(t)+y(t)=t;\ \ y(0)=1, \dot{y}(0)=-1$

$$s^2Y(s)-sy(0)-\dot{y}(0)+Y(s)=\frac{1}{s^2}$$

$$Y(s)=\frac{s^3-s^2+1}{s^2\left(s^2+1\right)}=\frac{C_1}{s^2}+\frac{C_2}{s}+\frac{C_3s+C_4}{s^2+1}$$

$$C_1=\left[s^2Y(s)\right]_{s=0}=1$$

$$C_1\left(s^2+1\right)+C_2s\left(s^2+1\right)+s^2\left(C_3s+C_4\right)=s^3-s^2+1$$

$$s^3: C_2 + C_3 = 1$$
$$s^2: C_1 + C_4 = -1 \quad \text{therefore:} \quad C_3 = 1, \quad C_4 = -2, \quad C_2 = 0$$
$$s: C_2 = 0$$

$$Y(s) = \frac{1}{s^2} + \frac{s-2}{s^2+1} = \frac{1}{s^2} + \frac{s}{s^2+1} - 2\frac{1}{s^2+1}$$

$$\boxed{y(t) = r(t) + \cos t - 2\sin t = (t + \cos t - 2\sin t)\cdot 1(t)}$$