ENGG407-Solution for Problem Set #1
Taylor Series, Numerical Errors and Numerical Derivatives
Winter 2010

Answer 1

a)

$$f(0) = \ln(0+1) = 0$$

$$f^{(1)}(x) = \frac{1}{x+1} \qquad \Rightarrow f^{(1)}(0) = \frac{1}{0+1} = 1$$

$$f^{(2)}(x) = -\frac{1}{(x+1)^2} \qquad \Rightarrow f^{(2)}(0) = -\frac{1}{(0+1)^2} = -1$$

$$f^{(3)}(x) = \frac{2}{(x+1)^3} \qquad \Rightarrow f^{(3)}(0) = \frac{2}{(0+1)^3} = 2$$

$$f(x) \approx f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$f(x) \approx x - \frac{x^2}{2} \text{ or } f(x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$$

b)

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$$

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{(0+1)^n}\frac{(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}$$

$$f(x) \approx x - \frac{x^2}{2} + \ldots + \frac{(-1)^{n-1}}{n}x^n$$

Answer 2

Part a:

$$f(1) = 2e^1 + 3\sin(1)$$

$$f'(1) = 2e^1 + 3\cos(1)$$

$$f^{(2)}(1) = 2e^1 - 3\sin(1)$$

$$f(x) \approx f(1) + f'(1)(x-1) + \frac{1}{2}f^{(2)}(1)(x-1)^2$$

Part b: To determine this polynomial it is necessary to expand the 3rd order Taylor series about $x_0 = 1$. Therefore

$$f(x) \approx f(1) + f'(1)(x-1) + \frac{1}{2} f^{(2)}(1)(x-1)^2 + f^{(3)}(1)\frac{(x-1)^3}{6}$$

where

$$f(1) = 2e^1 + 3\sin(1)$$

$$f^{(1)}(x_0) = 2e^{x_0} + 3\cos(x_0) = 2e^1 + 3\cos(1)$$

$$f^{(2)}(x_0) = 2e^{x_0} - 3\sin(x_0) = 2e^1 - 3\sin(1)$$

$$f^{(3)}(x_0) = 2e^{x_0} - 3\cos(x_0) = 2e^1 - 3\cos(1)$$

This results in a 3rd order polynomial in terms of x.

Part C:

$$e \approx \left| \frac{f^{(8)}(1)}{8!} \right| (x-1)^8$$

$$e \approx \left| \frac{2e^1 + 3\sin(1)}{8!} \right| (x-1)^8$$

Answer 3

Taylor series expansion about x = 1
Second order expansion is required since the output must be a quadratic

$$f_a(x) = xe^x\Big|_{x=1} \quad + \quad (1+x)e^x\Big|_{x=1}.(x-1) \quad + \quad \frac{1}{2}(2+x)e^x\Big|_{x=1}.(x-1)^2$$

$$= 1e^1 + 2e^1(x-1) + \frac{3}{2}e^1(x-1)^2$$

$$= e^1 + 2xe^1 - 2\,e^1 + 1.5\,e^1x^2 - 3e^1x + 1.5\,e^1$$

$$= 0.5\,e^1 - e^1x + 1.5\,e^1x^2$$

The polynomial parameters are:

A = 1.5e^1
B = -e^1
C = 0.5e^1

The approximation to f(x) at x = 0 is f(0) ≈ 0.5e^1

The true error is Et = |0 – 0.5e^1 |

The relative error is ε_t = |0 – 0.5e^1 | / 0

Answer 4

a) The Jacobian

$$J(x,y)=\begin{bmatrix}\dfrac{\partial f(x,y)}{\partial x}\\ \dfrac{\partial f(x,y)}{\partial y}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial \sin(xy)}{\partial x}\\ \dfrac{\partial \sin(xy)}{\partial y}\end{bmatrix}=\begin{bmatrix}y\cos(xy)\\ x\cos(xy)\end{bmatrix}$$

b) The Hessian

$$H(x,y)=\begin{cases}\dfrac{\partial J}{\partial x}\\ \dfrac{\partial J}{\partial y}\end{cases}\rightarrow\begin{cases}\dfrac{\partial J}{\partial x}=\begin{pmatrix}-y^2\sin(xy)\\ \cos(xy)-xy\sin(xy)\end{pmatrix}\\ \dfrac{\partial J}{\partial y}=\begin{pmatrix}\cos(xy)-xy\sin(xy)\\ -x^2\sin(xy)\end{pmatrix}\end{cases}=\begin{bmatrix}-y^2\sin(xy) & \cos(xy)-xy\sin(xy)\\ \cos(xy)-xy\sin(xy) & -x^2\sin(xy)\end{bmatrix}$$

c) The second order Taylor series of the function $f(x,y)=\sin(x,y)$ at expansion point $X=\begin{bmatrix}x_0\\ y_0\end{bmatrix}$

$$J(0,0)=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$H(0,0)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$f(0,0)=0$$

$$\tilde{f}_1(x,y)=f(x_0,y_0)+\begin{bmatrix}x-x_0 & y-y_0\end{bmatrix}J(x_0,y_0)+\frac{1}{2}\begin{bmatrix}x-x_0 & y-y_0\end{bmatrix}H(x_0,y_0)\begin{bmatrix}x-x_0\\ y-y_0\end{bmatrix}$$

$$=0+\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}x & y\end{bmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

Answer 5

4.3 (a) For this case $x_i = 0$ and $h = x$. Thus,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \cdots$$

$$f(0) = f'(0) = f''(0) = e^0 = 1$$

$$f(x) = 1 + x + \frac{x^2}{2} + \cdots$$

(b)

$$f(x_{i+1}) = e^{-x_i} - e^{-x_i}h + e^{-x_i}\frac{h^2}{2} - e^{-x_i}\frac{h^3}{6} + \cdots$$

for $x_i = 0.2$, $x_{i+1} = 1$ and $h = 0.8$. True value = $e^{-1} = 0.367879$.

zero order:

$$f(1) = e^{-0.2} = 0.818731 \qquad \varepsilon_t = \left|\frac{0.367879 - 0.818731}{0.367879}\right| \times 100\% = 122.55\%$$

first order:

$$f(1) = 0.818731 - 0.818731(0.8) = 0.163746 \qquad \varepsilon_t = \left|\frac{0.367879 - 0.163746}{0.367879}\right| \times 100\% = 55.49\%$$

second order:

$$f(1) = 0.818731 - 0.818731(0.8) + 0.818731\frac{0.8^2}{2} = 0.42574 \quad \varepsilon_t = \left|\frac{0.367879 - 0.42574}{0.367879}\right| \times 100\% = 15.73\%$$

third order:

$$f(1) = 0.818731 - 0.818731(0.8) + 0.818731\frac{0.8^2}{2} - 0.818731\frac{0.8^3}{6} = 0.355875$$

$$\varepsilon_t = \left|\frac{0.367879 - 0.355875}{0.367879}\right| \times 100\% = 3.26\%$$

Answer 6

4.5 True value: $f(3) = 554$.

<u>zero order:</u>

$$f(3) = f(1) = -62 \qquad \varepsilon_t = \left|\frac{554-(-62)}{554}\right| \times 100\% = 111.191\%$$

<u>first order:</u>

$$f(3) = -62 + f'(1)(3-1) = -62 + 70(2) = 78 \qquad \varepsilon_t = 85.921\%$$

<u>second order:</u>

$$f(3) = 78 + \frac{f''(1)}{2}(3-1)^2 = 78 + \frac{138}{2}4 = 354 \qquad \varepsilon_t = 36.101\%$$

<u>third order:</u>

$$f(3) = 354 + \frac{f^{(3)}(1)}{6}(3-1)^3 = 354 + \frac{150}{6}8 = 554 \qquad \varepsilon_t = 0\%$$

Thus, the third-order result is perfect because the original function is a third-order polynomial.

Answer 7

4.6 True value:

$$f'(x) = 75x^2 - 12x + 7$$

$$f'(2) = 75(2)^2 - 12(2) + 7 = 283$$

function values:

$x_{i-1} = 1.8$	$f(x_{i-1}) = 50.96$
$x_i = 2$	$f(x_i) = 102$
$x_{i+1} = 2.2$	$f(x_{i+1}) = 164.56$

forward:

$$f'(2) = \frac{164.56 - 102}{0.2} = 312.8 \qquad \varepsilon_t = \left|\frac{283 - 312.8}{283}\right| \times 100\% = 10.53\%$$

backward:

$$f'(2) = \frac{102 - 50.96}{0.2} = 255.2 \qquad \varepsilon_t = \left|\frac{283 - 255.2}{283}\right| \times 100\% = 9.823\%$$

centered:

$$f'(2) = \frac{164.56 - 50.96}{2(0.2)} = 284 \qquad \varepsilon_t = \left|\frac{283 - 284}{283}\right| \times 100\% = 0.353\%$$

Answer 8

4.7 True value:

$$f''(x) = 150x - 12$$
$$f''(2) = 150(2) - 12 = 288$$

$\underline{h = 0.25:}$

$$f''(2) = \frac{f(2.25) - 2f(2) + f(1.75)}{0.25^2} = \frac{182.1406 - 2(102) + 39.85938}{0.25^2} = 288$$

$\underline{h = 0.125:}$

$$f''(2) = \frac{f(2.125) - 2f(2) + f(1.875)}{0.125^2} = \frac{139.6738 - 2(102) + 68.82617}{0.125^2} = 288$$

Both results are exact because the errors are a function of 4^{th} and higher derivatives which are zero for a 3^{rd}-order polynomial.

Answer 9

4.14

$$f(x) = ax^2 + bx + c$$
$$f'(x) = 2ax + b$$
$$f''(x) = 2a$$

Second order Taylor Series Approximation Formula

Substitute these relationships into Eq. (4.4),

$$ax_{i+1}^2 + bx_{i+1} + c = ax_i^2 + bx_i + c + (2ax_i + b)(x_{i+1} - x_i) + \frac{2a}{2!}(x_{i+1}^2 - 2x_{i+1}x_i + x_i^2)$$

Collect terms

$$ax_{i+1}^2 + bx_{i+1} + c = ax_i^2 + 2ax_i(x_{i+1} - x_i) + a(x_{i+1}^2 - 2x_{i+1}x_i + x_i^2) + bx_i + b(x_{i+1} - x_i) + c$$
$$ax_{i+1}^2 + bx_{i+1} + c = ax_i^2 + 2ax_ix_{i+1} - 2ax_i^2 + ax_{i+1}^2 - 2ax_{i+1}x_i + ax_i^2 + bx_i + bx_{i+1} - bx_i + c$$
$$ax_{i+1}^2 + bx_{i+1} + c = (ax_i^2 - 2ax_i^2 + ax_i^2) + ax_{i+1}^2 + (2ax_ix_{i+1} - 2ax_{i+1}x_i) + (bx_i - bx_i) + bx_{i+1} + c$$
$$ax_{i+1}^2 + bx_{i+1} + c = ax_{i+1}^2 + bx_{i+1} + c$$

Answer 10

If you have solved this problem please discuss your results with your instructor.