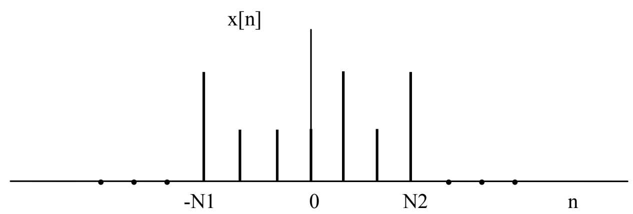
Chapter 5
Discrete Time Fourier Transform

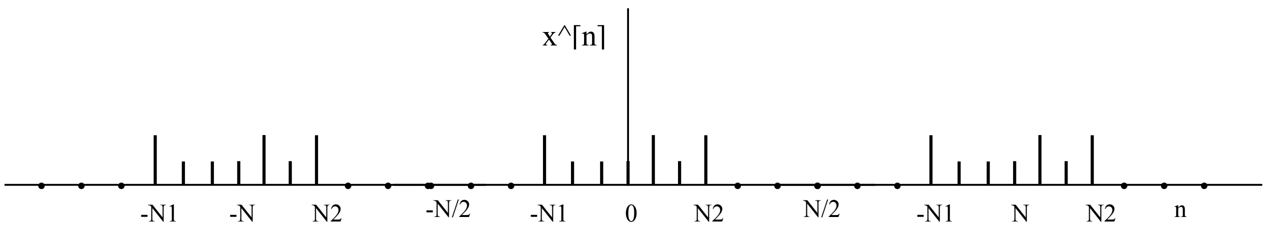
DT Fourier Transform of Aperiodic Signals:

Consider a general aperiodic signal x[n] given as:



$$x[n] = 0, \text{ outside } \quad -N1 \le n \le N2.$$

Now consider a periodic signal x^[n] of which x[n] is its one period as shown.



As x^[n] is periodic,

$$x\hat{}[n] = \sum_{k=<N>} a_k \, e^{jk\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N}. \qquad \text{... (1)}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x\hat{}[n] e^{-jk\omega_0 n} \qquad \text{...(2)}$$

In the interval -N/2 to N/2, x^[n]=x[n], hence

Thus

$$a_k = \frac{1}{N}\sum_{n=-N/2}^{N/2} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\omega_0 n} \quad \text{... (3)}$$

Defining

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad \text{... (4)}$$

eqn. (3) may be rewritten as:

$$a_k = \frac{1}{N} X(e^{jk\omega_0}) \quad \text{... (5)}$$

Replacing a_k in eqn.(1) by eqn (5):

$$x\hat{}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jk\omega_0})\, e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi}\sum_{k=<N>} X(e^{jk\omega_0})\, e^{jk\omega_0 n} .\, \omega_0 \quad \text{... (6)}$$

As $N \rightarrow \infty$, $\omega_0 \rightarrow 0$, summation becomes integral and x^[n] becomes x[n]. Hence,

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \text{... (7)}$$

$X[e^{j\omega}]$ is the Fourier transform of x[n].

Convergence of DT Fourier Transform:

It can be shown from eqn. (4) that Fourier transform exists if:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty,$$

or if :

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty.$$

Examples:

Problem 5.1 p.400 of text.

Find the FT of:

$$\text{a)} \quad (\frac{1}{2})^{n-1} u(n-1), \qquad \text{b)} \quad (\frac{1}{2})^{|n-1|} \quad .$$

Solution:

a)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (\frac{1}{2})^{n-1} u(n-1)\, e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} e^{-j\omega n} = \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} e^{-j\omega(n-1)} . e^{-j\omega}$$

$$= e^{-j\omega} \sum_{m=0}^{\infty} (\frac{1}{2})^{m} e^{-j\omega m} = e^{-j\omega} \sum_{m=0}^{\infty} (\frac{1}{2} e^{-j\omega})^{m}$$

$$= e^{-j\omega} \frac{1}{1 - \frac{1}{2} e^{-j\omega}}.$$

b)

$$x[n] = \frac{1}{2}^{|n-1|}.$$

$$n > 1: \quad x[n] = (\frac{1}{2})^{n-1}, \quad \text{let} \quad x_1 = (\frac{1}{2})^{n-1} u(n - 1).$$

$$n \le 0: \quad x[n] = (\frac{1}{2})^{-(n-1)}, \quad \text{let} \quad x_2 = (\frac{1}{2})^{-(n-1)} u(-n).$$

$$x[n] = x_1[n] + x_2[n]$$

$$X(e^{j\omega}) = \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} e^{-j\omega n} + \sum_{n=0}^{-\infty} (\frac{1}{2})^{-(n-1)} e^{-j\omega n}$$

Now,

$$\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} e^{-j\omega n} = e^{-j\omega} . \frac{1}{1 - \frac{1}{2} e^{-j\omega}}, \qquad \text{(done previously).}$$

Now,
$$\sum_{n=1}^{-\infty} (\frac{1}{2})^{-(n-1)} e^{-j\omega n} = \frac{1}{2} \sum_{n=0}^{-\infty} (\frac{1}{2} e^{j\omega})^{-n}$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} (\frac{1}{2} e^{j\omega})^{m} = \frac{1}{2} \frac{1}{1 - \frac{1}{2} e^{j\omega}}.$$

Hence,

$$X(e^{j\omega}) = e^{-j\omega} \frac{1}{1 - \frac{1}{2} e^{-j\omega}} + \frac{1}{2} \frac{1}{1 - \frac{1}{2} e^{j\omega}}.$$

Go through Examples 5.2 and 5.3 pp. 364-365 of text.

DT Fourier Transform of Periodic Signals:

- Can be shown that if x[n] with a period of N and expressed by the FS:

$$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N} \quad ,$$

then its Fourier transform is:

$$x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\omega_0) \qquad \text{... (5)}$$

- Eqn (5) is same as in CT periodic functions.

Go through Examples 5.5 and 5.6 pp. 371-372 of text.

Examples:

Problem 5.3 p. 400 of text.

Find the FT for $-\pi \le \omega < \pi$ of

a) $\sin(\frac{\pi}{3}n + \frac{\pi}{4})$, b) $2 + \cos(\frac{\pi}{6}n + \frac{\pi}{8})$.

Solution:

a) $\sin(\frac{\pi}{3}n + \frac{\pi}{4}) = \sin(\frac{2\pi}{6}n + \frac{\pi}{4}) \;\Rightarrow\; N = 6,\ \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}.$

$$\sin(\frac{\pi}{3}n + \frac{\pi}{4}) = \frac{1}{2j}[e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})}]$$

$$= \frac{1}{2j} e^{j\frac{\pi}{4}} . e^{j\frac{\pi}{3}n} - \frac{1}{2j} e^{-j\frac{\pi}{4}} . e^{-j\frac{\pi}{3}n} = a_1\, e^{j\frac{\pi}{3}n} + a_2\, e^{-j\frac{\pi}{3}n}$$

where,

$$a_1 = \frac{1}{2j} e^{j\frac{\pi}{4}}, \quad a_{-1} = -\frac{1}{2j} e^{-j\frac{\pi}{4}}.$$

All other coefficients are zero.

Using eqn (5):

$$X(e^{j\omega}) = 2\pi a_1 \, \delta(\omega - \omega_0) + 2\pi a_{-1} \, \delta(\omega + \omega_0)$$

$$= \frac{\pi}{j} e^{j\frac{\pi}{4}} \, \delta(\omega - \frac{\pi}{3}) - \frac{\pi}{j} e^{-j\frac{\pi}{4}} \, \delta(\omega + \frac{\pi}{3}).$$

b)

$$x[n] = 2 + \cos(\frac{\pi}{6} n + \frac{\pi}{8}) = 2 + \cos(\frac{2\pi}{12} n + \frac{\pi}{8}), \quad N = 12, \; \omega_0 = \frac{\pi}{6}.$$

$$x[n] = 2 + \frac{1}{2}[e^{j(\frac{\pi}{6}n + \frac{\pi}{8})} + e^{-j(\frac{\pi}{6}n + \frac{\pi}{8})}]$$

$$= 2 + (\frac{1}{2} e^{j\frac{\pi}{8}}) e^{j\omega_0 n} + (\frac{1}{2} e^{-j\frac{\pi}{8}}) e^{-j\omega_0 n}.$$

Whence

$$a_0 = 0, \quad a_1 = \frac{1}{2} e^{j\frac{\pi}{8}}, \quad a_{-1} = \frac{1}{2} e^{-j\frac{\pi}{8}}.$$

All other coefficients are zero.

$$X(e^{j\omega}) = 2\pi \times 2\delta(\omega) + \frac{2\pi}{2} e^{j\frac{\pi}{8}} \, \delta(\omega - \omega_0) + \frac{2\pi}{2} e^{-j\frac{\pi}{8}} \, \delta(\omega + \omega_0)$$

$$= 4\pi \, \delta(\omega) + \pi[e^{j\frac{\pi}{8}} \, \delta(\omega - \frac{\pi}{6}) + e^{-j\frac{\pi}{8}} \, \delta(\omega + \frac{\pi}{6})]$$

Properties of DT Fourier Transforms:

Important properties:

Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

Linearity:

If $\quad x_1[n] \quad \rightarrow \quad X_1(e^{j\omega})$

$\quad\quad x_2[n] \quad \rightarrow \quad X_2(e^{j\omega}), \quad$ then

$$ax_1[n] + bx_2[n] \quad \rightarrow \quad aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Time Shifting & Frequency Shifting:

$$x[n] \quad \rightarrow \quad X(e^{j\omega})$$

$$x[n-n_0] \quad \rightarrow \quad e^{-j\omega n_0}\, X(e^{j\omega}),$$

$$e^{j\omega_0 n}\, x[n] \quad \rightarrow \quad X(e^{j(\omega-\omega_0)})$$

Conjugate & Conjugate Symmetry:

$$x[n] \quad \rightarrow \quad X(e^{j\omega})$$

$$x^*[n] \quad \rightarrow \quad X^*(e^{-j\omega}).$$

For real x[n]:

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$Ev\{x[n]\} \quad \rightarrow \quad Re\{X(e^{j\omega})\}$$

$$Od\{x[n]\} \quad \rightarrow \quad j.\, Im\{X(e^{j\omega})\}$$

Time Reversal:

$$x[-n] \quad \rightarrow \quad X(e^{-j\omega})$$

Time Expansion:

$$x[\frac{n}{k}] \quad \rightarrow \quad X(e^{jk\omega}), \qquad \frac{n}{k} \rightarrow \text{integer.}$$

Differentiation:

$$nx[n] \quad \rightarrow \quad j\frac{dX(e^{j\omega})}{d\omega}$$

Parseval's Relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Convolution:

$$x[n] * h[n] \quad \rightarrow \quad X(e^{j\omega}) . H(e^{j\omega})$$

Go through Examples 5.8, 5.9, 5.11, 5.12 and 5.13 of text.

Examples:

Problem 5.6 p.400 of text.

Given $x[n]$, express the FT of the following in terms of $X(e^{j\omega})$.

a) $x_1[n]= x[1-n]+x[-1-n]$,

c) $x_3[n]= (n-1)^2 x[n]$.

Solution:

a) $x[-n] \quad \rightarrow \quad X(e^{-j\omega})$

$$x[n-n_0] \rightarrow e^{-j\omega n_0} X(e^{j\omega})$$

$$x[1-n] = x[-n-(-1)] \quad \rightarrow \quad e^{j\omega}X(e^{-j\omega})$$

$$x[-n-1] \quad \rightarrow \quad e^{-j\omega}X(e^{-j\omega})$$

Hence,

$$x_1[n] = x[1-n] + x[-1-n] = (e^{j\omega} + e^{-j\omega})X(e^{-j\omega}) = 2\cos\omega.\ X(e^{-j\omega}).$$

c)

$$(n-1)^2 x[n] = n^2 x[n] - 2nx[n] + x[n]$$

$$nx[n] \quad \rightarrow \quad j\frac{d}{d\omega}X(e^{j\omega})$$

$$n^2 x[n] = n\{nx[n]\} \quad \rightarrow \quad -\frac{d^2}{d\omega^2}X(e^{j\omega}).$$

Thus,

$$x_3(e^{j\omega}) = -\frac{d^2}{d\omega^2}X(e^{j\omega}) - 2\,j\frac{d}{d\omega}X(e^{j\omega}) + X(e^{j\omega}).$$

Go through Examples 5.19 and 5.20 pp.398-399 of text.

Problem 5.13 p.402.

Given a parallel system with

$$h_1[n] = (\frac{1}{3})^n u[n] \quad \text{and} \quad H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}}.$$

Find $h_{2[}n]$.

Solution:

Parallel system:

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega}).$$

$$h_1[n] = (\frac{1}{3})^n u[n], \Rightarrow \quad H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}.$$

$$H(e^{j\omega}) = \frac{-12+5e^{-j\omega}}{(e^{-j\omega}-4)(e^{-j\omega}-3)} = -\frac{2}{1-\frac{1}{4}e^{-j\omega}} + \frac{1}{1-\frac{1}{3}e^{-j\omega}}.$$

Hence, $\quad H_2(e^{j\omega}) = -\frac{2}{1-\frac{1}{4}e^{-j\omega}},$ giving

$$h_2[n] = -2(\frac{1}{4})^n u[n].$$

Problem 5.19 p.403:

Given an LTI system:

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$$

Find:

a) $H(e^{j\omega})$ of the system,

b) Impulse response h[n].

Solution:

a)FT of the governing equation:

$$Y[e^{j\omega}] - \frac{1}{6}e^{-j\omega}\,Y[e^{j\omega}] - \frac{1}{6}e^{-2j\omega}\,Y[e^{j\omega}] = X(e^{j\omega})$$

or $$[1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-2j\omega}]\,Y(e^{j\omega}) = X(e^{j\omega}).$$

Hence

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-2j\omega}}.$$

b)

$$H(e^{j\omega}) = -\frac{6}{e^{-2j\omega} + e^{-j\omega} - 6} = -\frac{6}{(e^{-j\omega} - 2)(e^{-j\omega} + 3)}$$

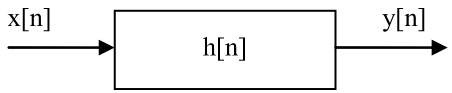
$$= \frac{3/5}{1 - \frac{1}{2}e^{-j\omega}} + \frac{2/5}{1 + \frac{1}{3}e^{-j\omega}}.$$

Hence,

$$h[n] = \frac{3}{5}(\frac{1}{2})^n\,u[n] + \frac{2}{5}(-\frac{1}{3})^n\,u[n].$$

Problem 5.20 p.403*:*

Given causal LTI system:



$$x[n] = (\frac{4}{5})^n\,u[n], \qquad y[n] = n\,(\frac{4}{5})^n\,u[n].$$

a) Find $H(e^{j\omega})$,

b) Write the system difference equation.

Solution:

a)

$$x[n] = (\frac{4}{5})^n \, u[n] \quad \Rightarrow \quad X(e^{j\omega}) = \frac{1}{1 - \frac{4}{5} e^{-j\omega}}.$$

$$y[n] = n\,(\frac{4}{5})^n \, u[n] \quad \Rightarrow \quad Y(e^{j\omega}) = j\frac{d}{d\omega}\{\text{FT of } (\frac{4}{5})^n \, u[n]\}$$

$$= j\frac{d}{d\omega}(\frac{1}{1 - \frac{4}{5} e^{-j\omega}}) = \frac{\frac{4}{5} e^{-j\omega}}{(1 - \frac{4}{5} e^{-j\omega})^2}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{4}{5} e^{-j\omega}}{1 - \frac{4}{5} e^{-j\omega}}.$$

b)

$$Y(e^{j\omega}) = H(e^{j\omega}).\, X(e^{j\omega}) = \frac{\frac{4}{5} e^{-j\omega}}{1 - \frac{4}{5} e^{-j\omega}} X(e^{j\omega})$$

or $$[1 - \frac{4}{5} e^{-j\omega}]\, Y(e^{j\omega}) = \frac{4}{5} e^{-j\omega}\, X(e^{j\omega})$$

Taking inverse FT:

$$y[n] - \frac{4}{5} y[n-1] = \frac{4}{5} x[n-1].$$

Problem 5.10 p.401 of text.

Given $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$, find the value of $A = \sum_{n=0}^{\infty} n(\frac{1}{2})^n$.

Solution:

$$A = \sum_{n=0}^{\infty} n(\frac{1}{2})^n = \sum_{n=-\infty}^{\infty} n(\frac{1}{2})^n \, u[n] = \sum_{n=-\infty}^{\infty} x[n]$$

where

$$x[n] = n(\frac{1}{2})^n \, u[n] = n \, x_1[n], \quad \text{where} \quad x_1[n] = (\frac{1}{2})^n \, u[n].$$

$$\text{Now,} \quad X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (\frac{1}{2})^n \, u[n] \, e^{-j\omega n} = \sum_{n=0}^{\infty} (\frac{1}{2})^n \, e^{-j\omega n}$$

$$= \frac{1}{1 - \frac{1}{2} e^{-j\omega}}.$$

Again,

$$x[n] = n \, x_1[n], \quad \text{hence}$$

$$X(e^{j\omega}) = j\frac{d}{d\omega} X_1(e^{j\omega}) = j\frac{d}{d\omega}\left(\frac{1}{1 - \frac{1}{2} e^{-j\omega}}\right) = \frac{\frac{1}{2} e^{-j\omega}}{(1 - \frac{1}{2} e^{-j\omega})^2}.$$

Now,

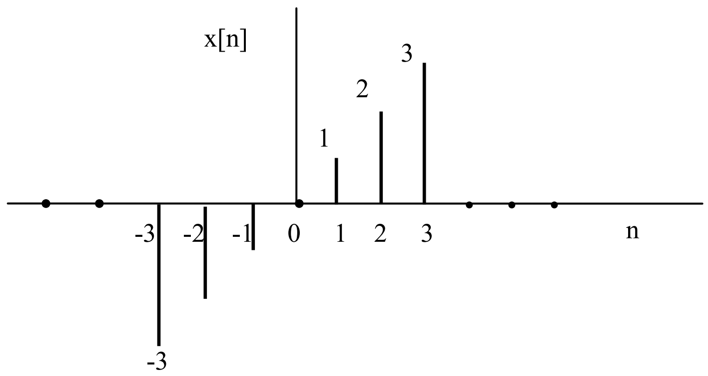
$$A = X(e^{j0}) = \frac{\frac{1}{2}}{(1 - \frac{1}{2})^2} = 2.$$

Problem 5.21 p.403 of text.

Find the FT of:

a) $x[n] = u[n-2] - u[n-6]$,

f)



j) $$x[n] = (n-1)\left(\frac{1}{3}\right)^{|n|}.$$

Solution:

a) $$x[n] = u[n-2] - u[n-6] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$$

$$X(e^{j\omega}) = e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega} + e^{-j5\omega}.$$

f) $$x[n] = -3\,\delta[n+3] - 2\,\delta[n+2] - \delta[n+1] + \delta[n-1] + 2\,\delta[n-2] + 3\,\delta[n-3]$$

$$X(e^{j\omega}) = -3e^{j3\omega} - 2e^{j2\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega}.$$

j)

$$x[n] = (n-1)\left(\frac{1}{3}\right)^{|n|}. \quad \text{Let} \quad x_1[n] = \left(\frac{1}{3}\right)^{|n|}.$$

$$X_1(e^{j\omega}) = \sum_{n=-\infty}^{-1} \left(\frac{1}{3}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n} e^{-j\omega n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n} e^{j\omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{3} e^{-j\omega}\right)^{n}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{3} e^{j\omega}\right)^{m+1} + \sum_{n=0}^{\infty} \left(\frac{1}{3} e^{-j\omega}\right)^{n}$$

$$= \frac{\frac{1}{3} e^{j\omega}}{1 - \frac{1}{3} e^{j\omega}} + \frac{1}{1 - \frac{1}{3} e^{-j\omega}} = \frac{4}{5 - 3\cos\omega}.$$

$$\text{Let} \quad y[n] = n\left(\frac{1}{3}\right)^{|n|} = n x_1[n],$$

$$Y(e^{j\omega}) = j\frac{d}{d\omega}[X_1(e^{j\omega})] = -j\frac{12\sin\omega}{(5 - 3\cos\omega)^2}.$$

$$\text{Now,} \quad x[n] = n\left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{3}\right)^{|n|}, \quad \text{thus}$$

$$X(e^{j\omega}) = \frac{4}{5 - 3\cos\omega} + j\frac{12\sin\omega}{(5 - 3\cos\omega)^2}.$$

Problem 5.22 p. 403 of text.

Find x[n], given

a) $$X(e^{j\omega}) = 1, \qquad \frac{\pi}{4} \le \omega \le \frac{3\pi}{4}$$

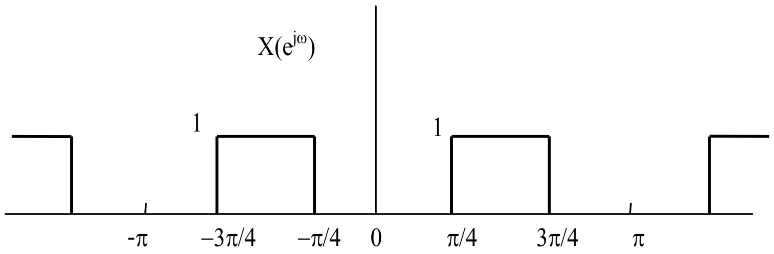
$$= 0, \qquad \text{else.}$$

d)

$$X(e^{j\omega}) = \cos^2 \omega + \sin^2 3\omega.$$

Solution:

a)



$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \left(\int_{-3\pi/4}^{-\pi/4} 1.\, e^{j\omega n}\, d\omega + \int_{\pi/4}^{3\pi/4} 1.\, e^{j\omega n}\, d\omega \right)$$

$$= \frac{1}{\pi n} \left(\sin \frac{3\pi}{4} n - \sin \frac{\pi}{4} n \right).$$

d)

$$X(e^{j\omega}) = \cos^2 \omega + \sin^2 3\omega = \frac{1 + \cos 2\omega}{2} + \frac{1 - \cos 3\omega}{2}$$

$$= 1 + \frac{1}{4} e^{j2\omega} + \frac{1}{4} e^{-j2\omega} - \frac{1}{4} e^{j3\omega} - \frac{1}{4} e^{-j3\omega}.$$

$$x[n] = \delta[n] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n+2] - \frac{1}{4}\delta[n-3] - \frac{1}{4}\delta[n+3].$$

Problem 5.33 p. 410 of text.

Given an LTI causal system: $y[n]+\frac{1}{2}y[n-1]=x[n]$

Find: (a) $H(e^{j\omega})$,

(b) the system response with:

ii) $x[n]=(-\frac{1}{2})^n\ u[n]$,

iii) $x[n]=\ \delta[n]+\frac{1}{2}\delta[n-1]$.

Solution:

(a) $$H(e^{j\omega})=\frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

(b) ii) $$X(e^{j\omega})=\frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega})=H(e^{j\omega}).X(e^{j\omega})=\frac{1}{(1+\frac{1}{2}e^{-j\omega})^2}.$$

$$\Downarrow$$

$$y[n]=(n+1)(-\frac{1}{2})^n\ u[n].$$

iii)

$X(e^{j\omega})=1+\frac{1}{2}e^{-j\omega}$, hence

$Y(e^{j\omega})=1.\quad\Rightarrow\quad y[n]=\delta[n].$

Problem 5.34 p. 410.

Given two LTI systems cascaded, with

$$H_1(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}},$$

$$H_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}.$$

Find: a) Difference equation of the overall system,

b) Impulse response of the overall system.

Solution:

(a)

$$H(e^{j\omega}) = H_1(e^{j\omega}) \,.\, H_2(e^{j\omega})$$

$$= \frac{2 - e^{-j\omega}}{(1 + \frac{1}{2}e^{-j\omega})(1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega})}$$

$$= \frac{2 - e^{-j\omega}}{1 + \frac{1}{8}e^{-j3\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}.$$

$$(1 + \frac{1}{8}e^{-j3\omega})Y(e^{j\omega}) = (2 - e^{-j\omega})\, X(e^{j\omega}),$$

$$\Downarrow$$

$$y[n] + \frac{1}{8}y[n-3] = 2x[n] - x[n-1].$$

b)

$$H(e^{j\omega}) = \frac{2 - e^{-j\omega}}{(1 + \frac{1}{2}e^{-j\omega})(1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega})}$$

$$= \frac{2 - e^{-j\omega}}{(1 + \frac{1}{2}e^{-j\omega})(e^{-j\omega} - 1 - \sqrt{3})(e^{-j\omega} - 1 + \sqrt{3})}$$

$$= \frac{1/3}{1 + \frac{1}{2}e^{-j\omega}} + \frac{1/(3 + 3\sqrt{3})}{-1 - \sqrt{3} + e^{-j\omega}} + \frac{1/(3 - 3\sqrt{3})}{-1 + \sqrt{3} + e^{-j\omega}}.$$

$$\Downarrow$$

$$h[n] = \frac{1}{3}(\frac{1}{2})^n u[n] + \frac{1}{(3 + 3\sqrt{2})}(-1 - \sqrt{3})^{n-1} u[n] + \frac{1}{(3 - 3\sqrt{2})}(-1 + \sqrt{3})^{n-1} u[n].$$