Chapter 3

Fourier Series Representation of Periodic Signals

If an arbitrary signal x(t) or x[n] is expressed as a linear combination of some basic signals, the response of an LTI system becomes the sum of the individual responses of those basic signals.

Such basic signal must:

- be capable of representing a large number of signals,
- have system responses simple enough for computational convenience.

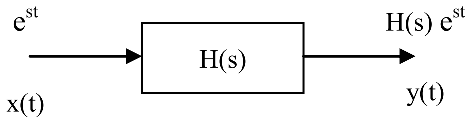
Complex exponentials, given below, are such basic signals/functions:

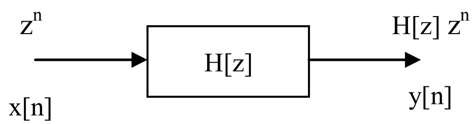
e^{st} for CT systems,

z^n for DT systems, where *s*, *z* are complex numbers.

Both have the property that:

- Response to an LTI system has the same form as the input with a change in the amplitude only.
- A function with this property is called *Eigen function* and the corresponding amplitude ratio is called *Eigen value*.





- e^{st} and z^n $\rightarrow$ Eigen functions,
- H(s) and H[z] $\rightarrow$ Eigen values.

Consider an input $x(t) = e^{st}$ in a CT system. The corresponding output:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau = H(s)\, e^{st}, \quad \text{where}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \qquad \dots \ (1)$$

Similarly, for a DT system:

Input: $x[n] = z^n$,

Output:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k].x[n-k] = \sum_{k=-\infty}^{\infty} h[k].z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k].z^{-k} = H[z]\, z^n,$$

where

$$H[z] = \sum_{k=-\infty}^{\infty} h[k]\ z^{-k} \qquad \text{... (2)}$$

CT systems:

LTI system. Thus, with an input $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$,

the output : $y(t) = a_1 H(s_1)\, e^{s_1 t} + a_2 H(s_2)\, e^{s_2 t} + a_3 H(s_3)\, e^{s_3 t}$.

In general: If $x(t) = \sum_k a_k\ e^{s_k t}$, ... (3)

then ... (4)

$$y(t) = \sum_k a_k H(s_k)\, e^{s_k t}.$$

DT systems:

If $x[n] = \sum_k a_k\ z_k{}^n$, ... (5)

then, $y[n] = \sum_k a_k\ H[z_k]\, z_k{}^n$ ... (6)

•If the input to an LTI system is a linear combination of complex exponentials, its output is also the linear combination of the same exponentials.

In Fourier series, such complex exponentials are:

$s = j\omega$, for CT systems

$z = e^{j\omega}$, for DT systems.

Fourier Series of CT Periodic Signals:

A signal *x(t)* with a fundamental frequency of ω_0 is expressed as a linear combination of complex exponential $e^{j\omega_0 t}$ and its harmonics $e^{\pm j2\omega_0 t}$, $e^{\pm j3\omega_0 t}$, .. as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \qquad \text{... (7)}$$

$k = 1$, fundamental frequency

$= 2$, second harmonics

$= 3$, third harmonics, so on.

R.H.S. Is called the '*Fourier series*' and a_k is called the *k*-th harmonic component.

Fourier Series Coefficient a_k *:*

From eqn. (7), we may write:

$$x(t)\, e^{-j\omega_0 nt} = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \, e^{-j\omega_0 nt} = \sum_{k=-\infty}^{\infty} a_k \, e^{j(k-n)\omega_0 t}$$

Integrating over an interval 0 to T (T is the fundamental period):

$$\int_0^T x(t)\, e^{-j\omega_0 nt}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k\, e^{j(k-n)\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \left(\int_0^T e^{j(k-n)\omega_0 t}\, dt\right) \qquad \text{... (8)}$$

Using Euler's formula:

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j\int_0^T \sin(k-n)\omega_0 t\, dt$$

$$= 0, \qquad \text{for } k \neq n$$

$$= T, \qquad \text{for } k = n\,.$$

Thus,

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = T, \qquad \text{for } k = n$$

$$= 0, \qquad \text{for } k \neq n \qquad \text{... (9)}$$

From eqn. (8):

$$\int_0^T x(t)\, e^{-j\omega_0 t}\, dt = a_n T\,.$$

or

$$a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \qquad \text{... (10a)}$$

As x(t) is periodic, eqn. (9) is applicable to any time interval T. Hence,

$$a_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt \qquad \text{.... (10 b)}$$

•Eqn. (7) is called *Fourier synthesis* equation and eqn. (10b) as *Fourier analysis* equation.

If x(t) is a **real** periodic function, then $x(t) = x^*(t)$, consequently

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{j\omega_0 kt} = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^*\, e^{-j\omega_0 kt} = \sum_{k=\infty}^{-\infty} a_{-k}^*\, e^{j\omega_0 kt}$$

$$= \sum_{k=-\infty}^{\infty} a_{-k}^*\, e^{j\omega_0 kt} \qquad \text{... (11)}$$

Comparing eqn. (7) with eqn. (11):

$$a_k^* = a_{-k}\,.$$

Problem 3.3 p. 251 of text.

Find fundamental frequency and Fourier co-efficient a_k of:

$$x(t) = 2 + \cos(\frac{2\pi}{3}t) + 4\sin(\frac{5\pi}{3}t).$$

Solution:

Using Euler's formula:

$$x(t) = 2 + \frac{1}{2}e^{j\frac{2\pi}{3}t} + \frac{1}{2}e^{-j\frac{2\pi}{3}t} + \frac{2}{j}e^{j\frac{5\pi}{3}t} - \frac{2}{j}e^{-j\frac{5\pi}{3}t}$$

$$= \sum_{k=-\infty}^{\infty} a_k \, e^{j\omega_0 kt}$$

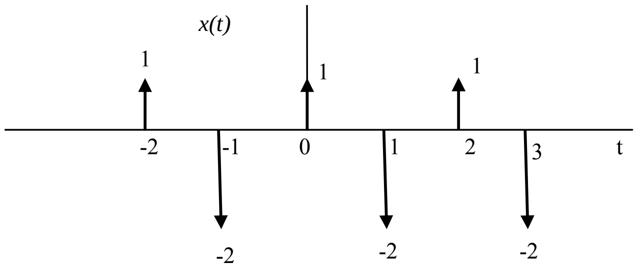
Hence, the fundamental frequency is $\omega_0 = \pi/3$. It consists of the fundamental, 2nd and 5th harmonics only.Thus,

$$x(t) = a_0 + a_2\, e^{j2\omega_0 t} + a_{-2}\, e^{-j2\omega_0 t} + a_5\, e^{j5\omega_0 t} + a_{-5}\, e^{-j5\omega_0 t}, \quad \text{giving}$$

$$a_0 = 2,\ a_2 = \frac{1}{2},\ a_{-2} = \frac{1}{2},\ a_5 = \frac{2}{j},\ a_{-5} = -\frac{2}{j}.$$

Problem 3.22 p. 256 of text.

d) Find the Fourier series of:



Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{j\omega_0 kt}$$

Here, $T = 2$, $\omega_0 = (2\pi/T) = \pi$.

$$a_k = \frac{1}{2}\int_0^2 x(t)\, e^{-j\omega_0 kt}\, dt = \frac{1}{2}[\int_0^1 \delta(t)\, e^{-j\pi kt}\, dt + \int_1^2 -2\,\delta(t-1)\, e^{-j\pi kt}\, dt]$$

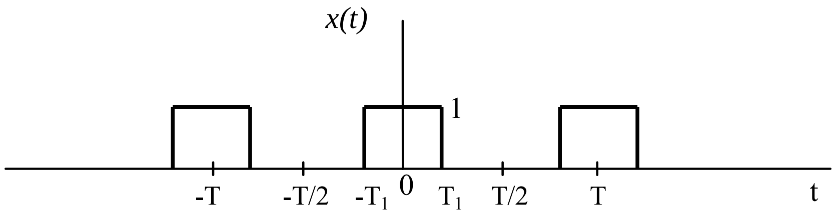
$$= \frac{1}{2}[\int_0^1 1.\ dt - 2\int_1^2 \ e^{-jk\pi}\ dt] = \frac{1}{2}[1 - 2\ e^{-jk\pi}] = \frac{1}{2}[1 - 2\ (e^{-j\pi})^k]$$

$$= \frac{1}{2} - (\cos\pi)^k = \frac{1}{2} - (-1)^k.$$

Problem 3.5 p. 193 of text. [Very important result]:

Find the Fourier series of:

Solution:



$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-j\omega_0 kt}\, dt = \frac{1}{T}.\frac{e^{-j\omega_0 kt}}{e^{-j\omega_0 k}}\Bigg|_{-T_1}^{T_1} = \frac{1}{-j\omega_0 kT}[\,e^{j\omega_0 kT_1} - e^{-j\omega_0 kT_1}\,]$$

$$= \frac{2}{\omega_0 kT}.\sin(\omega_0 kT_1) = \frac{\sin(\omega_0 kT_1)}{k\pi}.$$

Special case:

$$a_0 = \text{Lim}_{k \to 0}\, a_k = \frac{\omega_0 k T_1}{k\pi} = \frac{\frac{2\pi}{T}.kT_1}{k\pi} = \frac{2T_1}{T}.$$

Convergence of the Fourier Series:

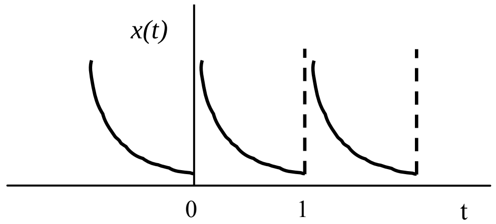
For a function to have Fourier series, the series is to be convergent. The convergence conditions for x(t) is given by *Dirichlet conditions* as follows:

Over any period, x(t):

i) must be absolutely integrable over any period, i.e.,

$$\int_T \left| x(t) \right| dt < \infty$$

Example:



The function has discontinuities and hence not convergent.

ii) does not have more than a finite number of discontinuities over any period.

iii) may have at most a finite number of finite discontinuities.

Gibbs Phenomenon:

• As the number of terms *N* is increased, Fourier series represents the function more accurately.

•At discontinuities, the series has a value ½ the sum of values just before and just after the discontinuity.

•The series shows ripples at discontinuity.

•As *N* increases, the ripples get compressed toward the discontinuity, but *its peak amplitude remained unchanged.*

Properties of Fourier Series:

- *Linearity:*

If $x(t) \rightarrow a_k$

$y(t) \rightarrow b_k$

Then

$$A\,x(t) + B\,y(t) \rightarrow A\,a_k + B\,b_k$$

•*Time Shift:*

$$x(t) \rightarrow a_k$$

$$x(t - t_0) \rightarrow e^{-jk\omega_0 t_0} . a_k$$

Time Reversal:

$$x(t) \rightarrow a_k$$
$$x(-t) \rightarrow a_{-k}$$

Time Scaling:

$$x(t) \rightarrow a_k$$
$$x(\alpha t) \rightarrow a_k \qquad \Rightarrow \; = \sum_{k=-\infty}^{\infty} a_k \, e^{jk(\alpha\omega_0)t}$$

Multiplication:

$$x(t) \rightarrow a_k$$
$$y(t) \rightarrow b_k$$
$$x(t).y(t) \rightarrow \sum_{l=-\infty}^{\infty} a_l \, . \, b_{k-l} \; .$$

Conjugate:

$$x(t) \rightarrow a_k$$
$$x^*(t) \rightarrow a_{-k}^*$$

Conjugate Symmetry:

For real x(t):

$$a_k = a_{-k}^*$$
$$a_k^* = a_{-k}$$

Derivative:

$$\frac{dx(t)}{dt} \rightarrow jk\omega_0 a_k$$

Parseval's Relation:

$$\frac{1}{T}\int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Consult Table 3.1 p. 206 of text.
- Go through example 3.8 p. 208 of text.

Examples:

Prob. 3.6 p.251 of text.

a)Which of the following are real valued?

b) Which are even?

$$x_1(t) = \sum_{k=0}^{100} (\frac{1}{2})^k \, e^{jk\frac{2\pi}{50}t}$$

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) \, e^{jk\frac{2\pi}{50}t}$$

$$x_3(t) = \sum_{k=-100}^{100} j\sin(\frac{k\pi}{2}) \, e^{jk\frac{2\pi}{50}t}$$

Solution:

a)

$$x_1(t) = \sum_{-\infty}^{\infty} a_k e^{j\omega_0 t}$$

$$x_1(t) \Rightarrow \quad a_k = (\frac{1}{2})^k, \qquad 0 \le k \le 100$$

$$= 0. \qquad \text{else.}$$

$$a_k = a_k^* = (\frac{1}{2})^{-k}, \qquad 0 \le k \le 100$$

$$= 0, \qquad \text{else.}$$

$a_k \neq a_{-k}^*$ $x_1(t)$ is not real-valued.

$$x_2(t): \quad a_k = \cos(k\pi), \qquad -100 \le k \le 100$$

$$= 0, \qquad \text{else.}$$

$$a_{-k} = \cos(k\pi), \qquad -100 \le k \le 100$$

$$= 0, \qquad \text{else}$$

$a_k = a_{-k}^*$. $x_2(t)$ is real.

$$x_3(t): \quad a_k = j\sin\frac{k\pi}{2}, \qquad -100 \le k \le 100$$

$$= 0. \qquad \text{else.}$$

$$a_{-k} = -j\sin\frac{k\pi}{2}, \qquad -100 \le k \le 100$$

$$= 0. \qquad \text{else.}$$

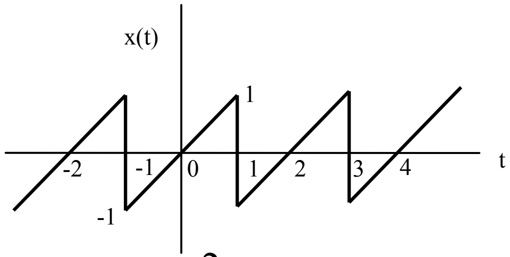
$$a_{-k}^* = j\sin\frac{k\pi}{2}, \qquad -100 \le k \le 100$$

$$= 0. \qquad \text{else.}$$

$a_k = a_{-k}^*$. $x_3(t)$ is real.

Problem 3.22(a) p.255 of text,

Find the Fourier series of x(t):



Solution:

$$x(t) = t, \quad \text{for} \quad -1 \le t \le 1; \ \text{and} \quad T = 2, \ \ \omega_0 = \frac{2\pi}{2} = \pi, \ \text{hence}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} a_k e^{j\pi kt}$$

where

$$a_k = \frac{1}{2}\int_{-1}^{1} t\, e^{-j\pi kt}\, dt = \frac{1}{2}\{ t. \frac{e^{-j\pi kt}}{-j\pi k}\Big|_{-1}^{1} - \int_{-1}^{1} 1. \frac{e^{-j\pi kt}}{-j\pi k} dt \}$$

$$= \frac{j}{2k\pi}[e^{jk\pi} + e^{-jk\pi}] + \frac{1}{2k^2\pi^2}[e^{-jk\pi} - e^{jk\pi}]$$

$$= \frac{j}{k\pi}\cos(k\pi) = \frac{j}{k\pi}\ (-1)^k.$$

Fourier Series Representation of DT Signals

DT Fourier Series:

As with CT signals, a DT signal can be expressed as a series combination of complex exponentials. The series, however, is *finite.*

Let x[n] be periodic with a fundamental period *N*, i.e.,

$$x[n]=x[n+N].$$

Consider a complex function

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(\frac{2\pi}{N})n}$$

For an integer *r* ,

$$\phi_{k+rN}[n] = e^{j(k+rN)(\frac{2\pi}{N})n} = e^{jk(\frac{2\pi}{N})n} . e^{jrN.(\frac{2\pi}{N})n}$$

$$= e^{jk(\frac{2\pi}{N})n} . e^{j2\pi rN} = e^{jk(\frac{2\pi}{N})n} = \phi_k[n].$$

or

$$\phi_k[n] = \phi_{k+rN}[n] \qquad \text{... (12)}$$

- $\phi_k[n]$ repeats itself with a period of N. Thus,
- a set of N values of $\phi_k[n]$ is enough to represent a periodic DT signal.

Therefore, a periodic DT signal may be expressed as

$$x[n] = \sum_{k=<N>} a_k \phi_k[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N}. \qquad \text{... (13)}$$

where <N> means *any* arbitrary consecutive N exponentials.

Eqn. (13) is the DT Fourier series, a_k is called Fourier coefficient.

Determination of a_k:

Consider

$$x[n]e^{-jr\omega_0 n} = \sum_{k=<N>} a_k e^{-j(k-r)\omega_0 n}$$

Now:

$$\sum_{n=<N>} x[n]e^{-jr\omega_0 n} = \sum_{n=<N>} \sum_{k=<N>} a_k e^{-j(k-r)\omega_0 n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{-j(k-r)\omega_0 n}$$

$$= \sum_{k=<N>} a_k \sum_{n=<N>} e^{-j(k-r)(\frac{2\pi}{N})n}$$

$$= N, \quad \text{if } k - r = 0, \pm N, \pm 2N\ldots$$

$$= 0, \quad \text{else.}$$

With $k = r$,

$$\sum_{n=<N>} x[n]e^{-jr\omega_0 n} = a_r N .$$

Replacing r by k ,

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N}. \qquad \text{... (14)}$$

Eqn. (13) is called *synthesis* equation and eqn. (14) the *analysis* equation.

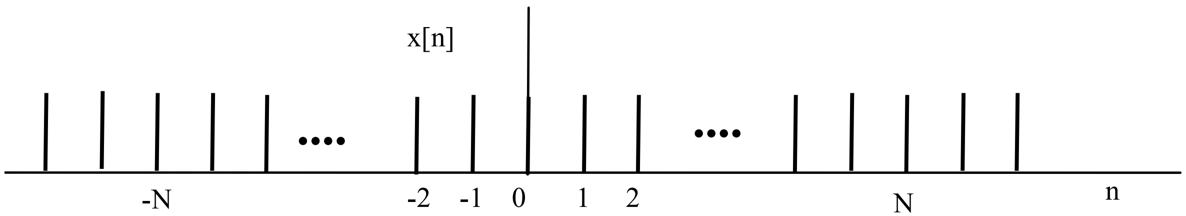
As a_k in an interval N repeats itself in any other periods,

$$a_k = a_{k \pm N} = a_{k \pm 2N} = \ldots$$

- Since the series is finite having N terms, it is always convergent.

Example:

Example 3.12 p.218 of text. Find the Fourier series of:



Here, N = 9.

$$a_k = \frac{1}{N} \sum_{n=-N1}^{N1} e^{-jk\omega_0 n} = \frac{1}{N} \sum_{m=0}^{2N1} e^{-jk\omega_0 (m-N1)} = \frac{1}{N} e^{jk\omega_0 N1} \sum_{m=0}^{2N1} e^{-jk\omega_0 m}$$

$$= \frac{1}{N} \frac{\sin k\omega_0 \frac{2N1+1}{2}}{\sin \frac{k\omega_0}{2}}.$$

For $k = 0$,

$$a_0 = \frac{1}{N}\frac{k\omega_0(2N1+1)}{\frac{k\omega_0}{2}} = \frac{1}{N}(2N1+1) = \frac{2N1+1}{N}$$

With N=9, there are 9 terms, e.g.,

$a_{-4}, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3,$ and a_4.

If the R.H.S. Of eqn. (14) is truncated to less than 9 terms, distortion occurs. See Fig. 3.18 p.220 of text.

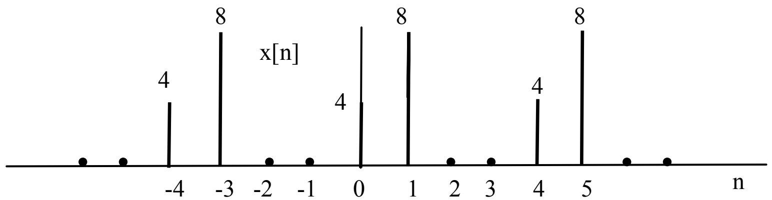
Note that Gibbs phenomenon does not occur in DT signal approximation.

Problem 3.9 p. 252 of text.

Find the Fourier series of

$$x[n] = \sum_{m=-\infty}^{\infty}\{4\delta[n-4m] + 8\delta[n-1-4m]\}$$

Solution:



Here,

$$N = 4, \qquad \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}.$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j\omega_0 kn} = \frac{1}{N}\sum_{n=0}^{3}\{4\delta[n] + 8\delta[n-1]\}e^{-j\omega_0 kn}$$

$$= \frac{1}{4}[4e^{-j\omega_0 kn.0} + 8e^{-j\omega_0 k.1}] = \frac{1}{4}[4 + 8e^{-j\omega_0 k}] = 1 + 2e^{-\frac{j\pi}{2}k}$$

giving

a_0=3, a_1 = 1-j2, a_2= -1, a_3= 1+j2.

$a_4 = a_0 = a_{-4}$ = ... = 3;

$a_5 = a_1 = a_{-3} = a_9$ = ... = 1-j2, etc.

Problem 3.30 p. 258 of text.

Find the Fourier coefficients of x[n], y[n] and z[n] given by:

$$x[n] = 1 + \cos(\frac{2\pi}{6}n)$$

$$y[n] = \sin(\frac{2\pi}{6}n + \frac{\pi}{4})$$

$$z[n] = x[n].\,y[n].$$

Solution:

N=6 for x[n], y[n] and hence for z[n].

$$\omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}.$$

a)

$$x[n]=1+\cos(\frac{2\pi}{6})n=1+\frac{1}{2}(e^{j\frac{\pi}{3}n}+e^{-j\frac{\pi}{3}n})=1+\frac{1}{2}(e^{j\omega_0 n}+e^{-j\omega_0 n})$$

$$=\sum_{k=<N>} a_k e^{j\omega_0 kn}.$$

Hence, $a_0=1,\quad a_1=\frac{1}{2},\quad a_{-1}=\frac{1}{2}.$

b)

$$y[n]=\sin(\frac{\pi}{3}n+\frac{\pi}{4})=\sin(\omega_0 n+\frac{\pi}{4})=\frac{1}{2j}[e^{j(\omega_0 n+\frac{\pi}{4})}-e^{-j(\omega_0 n+\frac{\pi}{4})}]$$

$$=\frac{e^{j\frac{\pi}{4}}}{2j}e^{j\omega_0 n}-\frac{e^{-j\frac{\pi}{4}}}{2j}e^{-j\omega_0 n},\quad \text{giving}$$

$$b_0=0,\quad b_1=\frac{e^{j\frac{\pi}{4}}}{2j},\quad b_{-1}=-\frac{e^{-j\frac{\pi}{4}}}{2j}.$$

(c)

$$x[n].y[n]\ \rightarrow\ c_k=\sum_{l=<N>} b_l a_{k-l}.$$

Choosing $<N>$ to cover b_{-1} and a_{-1},

$a_2=a_3=a_4=0,$ and $b_2=b_3=b_4=0.$

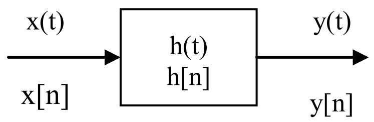
$$c_k=\sum_{-1}^{4} b_l a_{k-l}=b_{-1}a_{k+1}+b_0 a_k+b_1 a_{k-1}+0,\quad \text{giving}$$

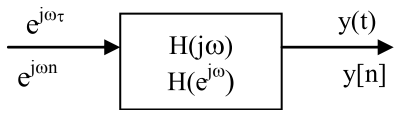
$$c_0=b_{-1}a_1+b_0 a_0+b_1 a_{-1}=\frac{1}{2}\frac{e^{j\frac{\pi}{4}}}{2j}+0-\frac{1}{2}\frac{e^{-j\frac{\pi}{4}}}{2j}=\frac{1}{2}\sin\frac{\pi}{4},$$

$$c_1=\frac{e^{j\frac{\pi}{4}}}{2j}=c_{-1}^*,\quad c_2=\frac{e^{j\frac{\pi}{4}}}{4j}=c_{-2}^*,\ c_3=c_{-3}^*=0.$$

Fourier Series and LTI Systems

For a system with an impulse response $H(j\omega)$ or $H[e^{j\omega}]$:





For CT:

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau \qquad \text{... (15a)}$$

For DT:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k} \qquad \text{... (15b)}$$

CT: With Fourier series of x(t):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \quad ,$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k\, H(jk\omega_0)\, e^{jk\omega_0 t} \qquad \text{... (16)}$$

DT:

$$y[n] = \sum_{k=<N>} a_k\, H(e^{j\omega_0 k})\, e^{j\omega_0 nk} \qquad \text{... (17)}$$

To find the system output y(t) or y[n]:

1.Find $H(j\omega)$ or $H(e^{jw})$ using eqn. 15a or 15b,

2. Find F. S. of x(t) or x[n].

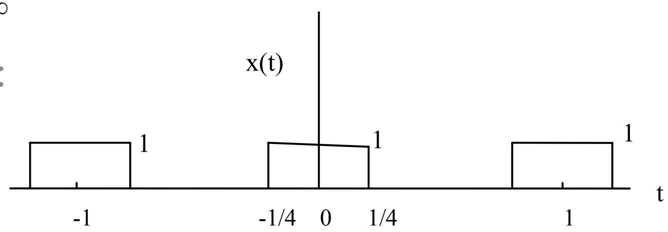
3.Find y(t) or y[n] using eqn. 16 or eqn.17.

Prob. 3.34 p. 260 of text.

Given $h(t)=e^{-4|t|}$, find $y(t)$ with

(a) $$x(t)=\sum_{n=-\infty}^{\infty}\delta(t-n)$$

(c) x(t) given as:



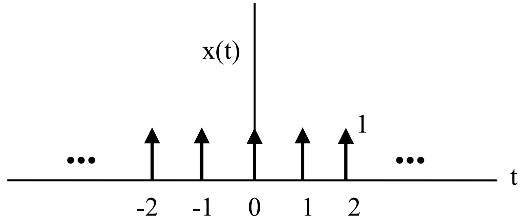
Solution:

$$H(j\omega)=\int_{-\infty}^{\infty}h(\tau)\,e^{-j\omega\tau}\,d\tau=\int_{-\infty}^{0}e^{4\tau}\,e^{-j\omega\tau}\,d\tau+\int_{0}^{\infty}e^{-4\tau}\,e^{-j\omega\tau}\,d\tau$$

$$=\frac{1}{4-j\omega}+\frac{1}{4+j\omega}.$$

(a)

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{j\omega_0 kt}\ .$$



Here, $T=1$, $\omega_0=2\pi$.

$$a_k=\frac{1}{T}\int_T\sum_{n=-\infty}^{\infty}\delta(t-n)\,e^{-j\omega_0 kt}\,dt=\frac{1}{T}\int_T\sum_{0}^{1}\delta(t)\,e^{-j\omega_0 k.0}\,dt$$

$$=\frac{1}{1}\int_0^1 1.dt=1$$

Hence,

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \, H(j\omega_0 k)\, e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} H(j2\pi k)\, e^{j2\pi kt}$$

$$= \sum_{k=-\infty}^{\infty} \left[\frac{1}{4 - j2\pi k} + \frac{1}{4 + j2\pi k}\right] e^{j2\pi kt}.$$

(c)
$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\omega_0 k T_1)}{k\pi} e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} \frac{\sin(\frac{\pi k}{2})}{k\pi} e^{j2\pi kt}.$$

$$y(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\frac{\pi k}{2})}{k\pi} \left[\frac{1}{4 - j2\pi k} + \frac{1}{4 + j2\pi k}\right] e^{j2\pi kt} = \sum_{k=-\infty}^{\infty} b_k \, e^{j2\pi kt}$$

where, $b_k = 0, \quad \text{for} \quad k \to \text{even}$

$$= \frac{1}{4}, \quad \text{for } k = 0$$

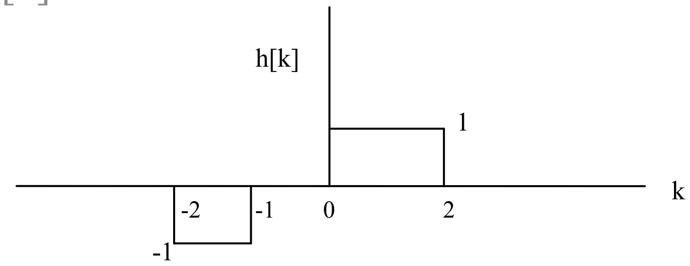
$$= \frac{\sin(\frac{\pi}{2} k)}{\pi k} \left[\frac{1}{4 - j2\pi k} + \frac{1}{4 + j2\pi k}\right], \quad \text{for } k \to \text{odd}.$$

Prob. 3.38 p. 261:

Given:
$$h[n] = 1, \qquad 0 \le n \le 2$$
$$= -1, \qquad -2 \le n \le -1$$
$$= 0. \qquad \text{else.}$$

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k].$$

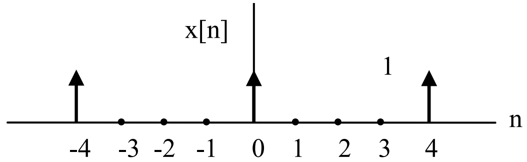
Find the Fourier coefficients of output y[n].



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}$$

$$= \sum_{k=-2}^{-1} (-1)\, e^{-j\omega k} + \sum_{k=0}^{2} 1.\, e^{-j\omega k}$$

$$= e^{-j\omega} - e^{j\omega} + 1 + e^{-j2\omega} - e^{j2\omega}$$



$$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{j\omega_0 n}, \text{ here } \quad N = 4, \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}.$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-j\omega_0 kn} = \frac{1}{4} \sum_{n=0}^{3} \delta[n]\, e^{-j\omega_0 kn} = \frac{1}{4}\, e^{-j\omega_0 k.0} = \frac{1}{4}.$$

$$y[n] = \sum_{k=0}^{3} a_k\, H(e^{j\omega_0 k})\, e^{j\omega_0 kn}$$

$$= \frac{1}{4} \sum_{k=0}^{3} [e^{-j\omega_0 k} - e^{j\omega_0 k} + 1 + e^{-j2\omega_0 k} - e^{j2\omega_0 k}] e^{j\omega_0 kn}$$

$$= \frac{1}{4} \sum_{k=0}^{3} b_k\, e^{j\omega_0 kn}$$

Where,

$$b_k = \frac{1}{4}[e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k} + 1 + e^{-j\pi k} - e^{j\pi k}]$$

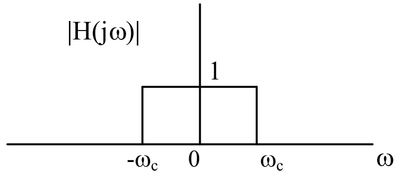
$$= \frac{1}{4}[1 + e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k}].$$

Frequency-Shaping or Frequency-Selective Filters:

Ideal Filters:

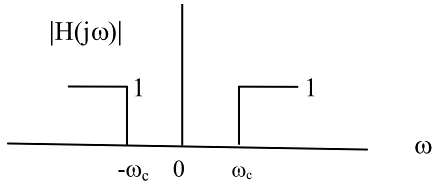
Low-pass:

$$H(j\omega) = 1, \qquad |\omega| < \omega_c$$
$$= 0, \qquad |\omega| > \omega_c$$



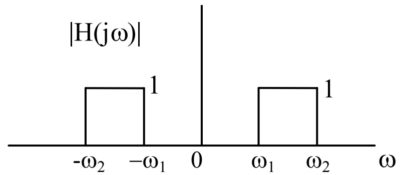
High-pass:

$$H(j\omega) = 0, \qquad |\omega| < \omega_c$$
$$= 1, \qquad |\omega| > \omega_c$$



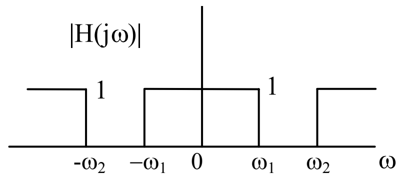
Band-pass:

$$H(j\omega) = 1, \qquad \omega_1 < |\omega| < \omega_2$$
$$= 0 \qquad \text{else.}$$



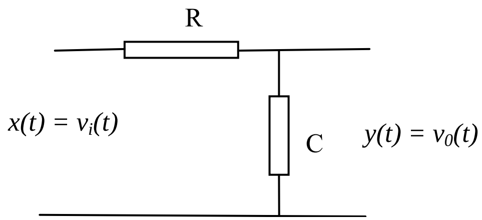
Band-stop:

$$H(j\omega) = 0, \qquad \omega_1 < |\omega| < \omega_2$$
$$= 1, \qquad \text{else.}$$



Non-ideal Filters:

CT Filter:



System differential equation:

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

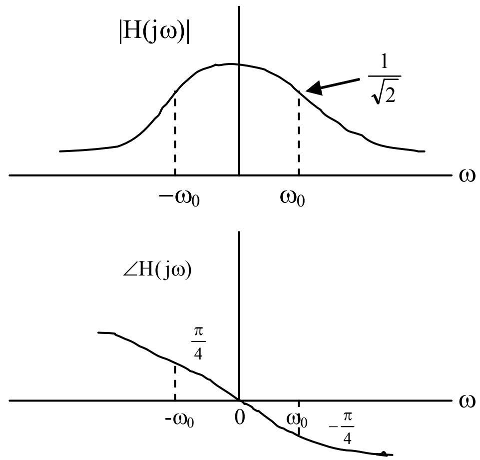
$$x(t) = e^{j\omega t}, \quad y(t) = H(j\omega)\, e^{j\omega t}, \quad \frac{dy(t)}{dt} = j\omega H(j\omega) e^{j\omega t}$$

or $(1 + j\omega RC)H(j\omega) = 1$, giving

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\frac{\omega}{\omega_0}}, \quad \omega_0 = \frac{1}{RC}$$

$$\left|H(j\omega)\right| = \frac{1}{\sqrt{1+(\frac{\omega}{\omega_0})^2}}, \quad \angle H(j\omega) = -\tan^{-1}\frac{\omega}{\omega_0}$$

The response plot:



This is a low-pass filter.

DT Filters

Consider a first-order filter given by the difference equation:

$$y[n] - a\, y[n-1] = x[n], \qquad |a| < 1.$$

With $x[n] = e^{j\omega n}$,

$$H(e^{j\omega}).\, e^{j\omega n} - a\, H(e^{j\omega})\, e^{j\omega(n-1)} = e^{j\omega n}, \text{ or}$$

$$H(e^{j\omega})[1 - ae^{-j\omega}]\, e^{j\omega n} = e^{j\omega n}, \quad \text{giving}$$

$$H(e^{j\omega}) = \frac{1}{1 - a\, e^{-j\omega}}.$$

The magnitude $|H(e^{j\omega})|$ and the phase angle $\angle H(e^{j\omega})$ for a=0.6 and a=-0.6 are shown in Fig, 3,34 p. 246 of the text.

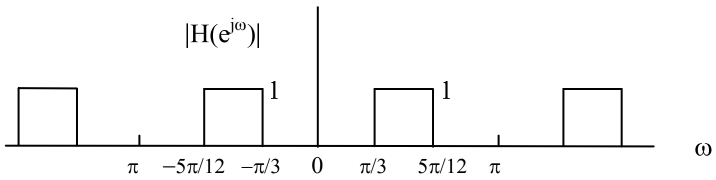
- For $0 < a < 1$, it is low-pass filter.
- For $0 > a > -1$, it is high-pass filter.

Problem 3.16 p. 253 of text.

With the given impulse response $H(e^{j\omega})$, find the filter output with the following inputs:

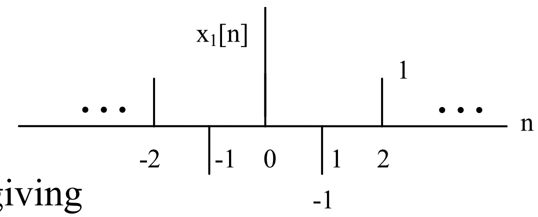
a) $x_1[n] = (-1)^n$

b) $x_2[n] = 1 + \sin(\frac{3\pi}{8} + \frac{\pi}{4})$



Solution:

a) $x_1[n] = (-1)^n$



Here, N=2 and $\omega_0=\pi$.

$$a_k = \frac{1}{2}\sum_{n=0}^{1}(-1)^n e^{-j\pi kn} = \frac{1}{2}(1 - e^{-j\pi k}),\ \text{giving}$$

$$a_0 = 0,\ a_1 = 1.$$

$$y[n] = \sum_{k=0}^{1} a_k\ H(e^{-j\pi k})\, e^{-j\pi kn} = a_1\ H(e^{j\pi}).\, e^{j\pi n}$$

$$= 0. \qquad (\text{since } H(e^{j\pi}) = 0).$$

None is passed.

b)

$$x_2[n] = 1 + \sin(\frac{3\pi}{8} + \frac{\pi}{4}) \Rightarrow \quad N = 16, \ \omega_0 = \frac{2\pi}{16} = \frac{\pi}{8}.$$

$$x_2[n] = 1 + \frac{1}{2j}[e^{j(\frac{3\pi}{8}+\frac{\pi}{4})} - e^{-j(\frac{3\pi}{8}+\frac{\pi}{4})}] = 1 + (\frac{1}{2j}e^{j\frac{\pi}{4}})e^{j\frac{3\pi}{8}} - (\frac{1}{2j}e^{-j\frac{\pi}{4}})e^{-j\frac{3\pi}{8}}$$

Hence, $\Rightarrow \quad a_0 = 1, \quad a_3 = \frac{1}{2j}e^{j\frac{\pi}{4}}, \quad a_{-3} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}, \ \text{others} = 0.$

$$y[n] = \sum_{k=<N>} a_k \, H(e^{j\omega_0 k})\, e^{j\omega_0 kn}$$

$$= a_0 \, H(e^{j0})\, e^{j0} + a_3 \, H(e^{j3\omega_0})\, e^{j3\omega_0 n} + a_{-3} \, H(e^{-j3\omega_0})\, e^{-j3\omega_0 n}$$

$$= 0 + a_3 \, H(e^{j\frac{3\pi}{8}})\, e^{j\frac{3\pi}{8}n} + a_{-3} \, H(e^{-j\frac{3\pi}{8}})\, e^{-j\frac{3\pi}{8}n}$$

$$= \frac{1}{2j}e^{\frac{j\pi}{4}}.1.e^{\frac{j3\pi}{8}n} - \frac{1}{2j}e^{-\frac{j\pi}{4}}.1.e^{-\frac{j3\pi}{8}n}$$

$$= \frac{1}{2j}[e^{j(\frac{3\pi}{8}n+\frac{\pi}{4})} - e^{-j(\frac{3\pi}{8}n+\frac{\pi}{4})}] = \sin(\frac{3\pi}{8}n + \frac{\pi}{4})$$

ω= 1 is blocked, ω= $3\pi/8$ is passed.