MATH3705A — Test 1:   17:35–18:25, June 2

Name and Student Number:

Total points: 30. No partial marks for Questions 1-4.
Closed book! Non-programmer calculators are allowed!

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[4] 1. $L\{t\cos(3t)\} =$

(a) $\frac{s^2-9}{(s^2+9)^2}$ (b) $\frac{3s}{s^2+9}$ (c) $\frac{3s^2+9}{(s^2+9)^2}$ (d) $-\frac{2e^s}{(s^2+3)^2}$ (e) $\frac{1}{s^2+9}$

Solution: (a)
$L\{\cos 3t\} = \frac{s}{s^2+9}, \Rightarrow L\{t\cos 3t\} = -\left(\frac{s}{s^2+9}\right)' = \frac{s^2-9}{(s^2+9)^2}$.

[4] 2. Let $H(s) = L\left\{\dfrac{\sin t}{t}\right\}$. Find $H(1)$.

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{5}$ (e) None of them

Solution: (c).
Since $L\{\sin t\} = \frac{1}{s^2+1}$, we imply that

$$H(s) = L\left\{\frac{\sin t}{t}\right\} = \int_s^{+\infty} \frac{1}{x^2+1}dx = \arctan x|_s^{+\infty} = \frac{\pi}{2} - \arctan(s).$$

$H(1) = \frac{\pi}{2} - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

[4] 3. Find $L\{e^{2t} * e^{3t}\}$.

(a) $\frac{1}{s-5}$ (b) $\frac{1}{s^2-5s+6}$ (c) $\frac{1}{s^2-5s-6}$ (d) $\frac{1}{s^2-6s+5}$ (e) $\frac{1}{s^2+6s-5}$

Solution: (b).

$$\mathcal{L}\{e^{2t} * e^{3t}\} = \mathcal{L}\{e^{2t}\}\mathcal{L}\{e^{3t}\} = \frac{1}{s-2}\frac{1}{s-3} = \frac{1}{s^2-5s+6}.$$

[4] 4. Calculate $L\left\{\dfrac{\sqrt{t}}{\sqrt{\pi}\, t}\right\}$.

(a) $s^{-1.5}$ (b) $s^{-0.5}$ (c) $s^{0.5}$ (d) $s^{1.5}$ (e) s

Solution: (b)

$$L\{\frac{\sqrt{t}}{t}\} = L\{t^{-0.5}\} = \frac{\Gamma(0.5)}{s^{0.5}} = \frac{\sqrt{\pi}}{s^{0.5}}.$$

[6] 5. Calculate $L\{u(t-2)t^3\}$.

Solution: We use the second shifting theorem:

$$\begin{aligned} L\{u(t-2)t^3\} &= L\{u(t-2)(t-2+2)^3\} \\ &= L\{u(t-2)[(t-2)^3+6(t-2)^2+12(t-2)+8]\} \qquad (2points) \\ &= e^{-2s}L\{t^3+6t^2+12t+8\} \qquad (2points) \\ &= e^{-2s}\{\frac{6}{s^4}+\frac{12}{s^3}+\frac{12}{s^2}+\frac{8}{s}\}. \qquad (2points) \end{aligned}$$

[8] 6. Using the Laplace transform solve IVP: $f''(t) + f'(t) - 6f(t) = 70e^{4t}$ with boundary conditions $f(0) = f'(0) = 0$.

Solution: Applying Laplace transform to the two sides, we get the equation

$$s^2F + sF - 6F = \frac{70}{s-4} \qquad (3points)$$

or

$$F(s) = \frac{70}{(s-4)(s+3)(s-2)}. \qquad (1point)$$

By partial fraction

$$\begin{aligned} \frac{70}{(s-4)(s+3)(s-2)} &= \frac{A}{s-4}+\frac{B}{s+3}+\frac{C}{s-2} \\ 70 &= A(s+3)(s-2)+B(s-4)(s-2)+C(s+3)(s-4). \end{aligned}$$

To find A, B, C, we use special s values: $s = 4$ gives $A = 5$, and $s = -3$ gives $B = 2$, and $s = 2$ gives $C = -7$. Putting all this together we have

$$F(s) = \frac{5}{s-4}+\frac{2}{s+3}+\frac{-7}{s-2}, \qquad (2points)$$

Thus

$$f(t) = 5e^{4t} + 2e^{-3t} - 7e^{2t}. \qquad (2points)$$

Table of Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\left\{t^p\right\} = \frac{\Gamma(p+1)}{s^{p+1}},\ p > -1$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha}F\left(\frac{s}{\alpha}\right), \quad \alpha > 0$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s), \quad s > a \geq 0$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0), \quad n \geq 0$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \equiv (-1)^n \frac{d^n}{ds^n}F(s), \quad n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)\,dx$$

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}\{f(t) * g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x)\,dx\right\} = F(s)G(s), \text{ where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}, \quad a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt \text{ if } f \text{ is periodic with period } T$$