MAT 2377, Probability and statistics for engineers

Assignment 2
Solutions

Solve the following exercises with a TI-30, TI-34, Casio FX-260 or Casio FX-300 calculator.

[5] 1. A ball bearing factory produces its product on three machines, A, B and C. Machine A produces 30% of the ball bearings, machine B produces 50% and machine C produces the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 2% from machine B and 4% from machine C.
(a) What fraction of total production is defective?
(b) A ball bearing is chosen at random from the production line and found to be defective. What is the probability that it came from
(i) machine A, (ii) machine B, (iii) machine C?

Solution: For a randomly selected ball bearing, write A, B, C, for the events that it was produced on machine A, B, C, respectively, and write D for the event that it is defective. Then the problem tells that events A, B, C are mutually exclusive and their union is the whole sample space (in normal language, a ball bearing from the factory is made on exactly one of machines A, B, C). Also it says that $P(D|A) = 5\% = 0.05$, etc.
(a) The questions asks for $P(D)$. By the rule of total probability, we have

$$\begin{aligned} P(D) &= P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) \\ &= (0.05)(0.3) + (0.02)(0.5) + (0.04)(0.2) \\ &= 0.033 \end{aligned}$$

(b) The question asks for:

$$\text{(i)} \quad P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{P(D|A)P(A)}{P(D)} = \frac{(0.05)(0.3)}{0.033} \sim 0.4545.$$

$$\text{(ii)} \quad P(B|D) = \frac{P(B \cap D)}{P(D)} = \frac{P(D|B)P(B)}{P(D)} = \frac{(0.02)(0.5)}{0.033} \sim 0.3030.$$

$$\text{(iii)} \quad P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{P(D|C)P(C)}{P(D)} = \frac{(0.04)(0.2)}{0.033} \sim 0.2424.$$

In each case we used Bayes.

[5] 2. Suppose that the probability mass function $f(x)$ for the discrete random variable X is given by the following table:

x	4	6	8	10
$f(x)$	0.4	0.3	0.2	0.1

(a) Find $P(X = 6)$.
(b) Find $P(X \leq 7)$.
(c) Find the mean of X.
(d) Find the standard devation of X.
(e) What is the expected value of $7 + 10X$?

Solution: (a) $P(X = 6) = f(6) = 0.3$.
(b) $P(X \leq 7) = f(4) + f(6) = 0.7$.
(c) $\mu_X = E[X] = 4(0.4) + 6(0.3) + 8(0.2) + 10(0.1) = 6$.
(d) We could use the definition $\sigma_X^2 = E[(X - \mu_X)^2]$, but here we will use the simpler formula $\sigma_X^2 = E[X^2] - \mu_X^2 = 4^2(0.4) + 6^2(0.3) + 8^2(0.2) + 10^2(0.1) - 6^2 = 4$ (both methods will give the same result). So the standard deviation is $\sigma_X = \sqrt{4} = 2$.
(e) By Theorem 2.5, we have $E[7 + 10X] = 7 + 10\,E[X] = 7 + 10\,(6) = 67$. Alternatively we could have calculated this directly.

[6] 3. The waiting time, in minutes, between successive speeders spotted by a radar unit is a continuous random variable X with probability density function

$$f(x) = \begin{cases} 0 & \text{if } x < 3, \\ \frac{81}{x^4} & \text{if } x \geq 3. \end{cases}$$

(a) Find the cumulative distribution function of X. Hint: it is 0 for $x < 3$.
(b) Find the probability of waiting more than 5 minutes between successive speeders
 (i) using the cumulative distribution function of X,
 (ii) using the probability density function of X.
(c) Find the expected waiting time between successive speeders.
(d) Find the standard deviation of X.

Solution: (a) We are asked for the cumulative distribution function F; it is given by

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} \displaystyle\int_{-\infty}^{x} 0\,dt = 0, & \text{if } x \leq 3 \\ \displaystyle\int_{3}^{x} \frac{81}{t^4}\,dt = \int_{3}^{x} 81t^{-4}\,dt = 1 - 27x^{-3}, & \text{if } x \geq 3 \end{cases}$$

(Note that since the limit on the integral is from 3 since $f(t)$ equals zero to the left of 3 and equals the formula $81/t^4$ to the right of 3.) So

$$F(x) = \begin{cases} 0, & x \leq 3 \\ 1 - \frac{27}{x^3}, & x \geq 3 \end{cases}$$

(b) We are asked for $P(X \geq 5)$. Note that $P(X = 5) = 0$ since X is a continuous random variable, so that the desired probability equals $P(X > 5)$.
(i) $P(X > 5) = 1 - P(X \leq 5) = 1 - F(5) = 1 - (1 - 27/5^3) = 27/5^3 = 0.216.$

(ii) $$P(X > 5) = \int_5^\infty f(x)\,dx = \int_5^\infty 81x^{-4}\,dx = -27x^{-3}\Big|_5^\infty = 27/5^3 = 0.216$$

Note that these two methods of calculation will have to give the same answer, as they calculate the same probability.
(c) We are asked for $E[X]$.

$$E[X] = \int_\infty^\infty xf(x)\,dx = \int_3^\infty x\frac{81}{x^4}\,dx = \int_3^\infty 81x^{-3} = -\frac{81}{2}x^{-2}\Big|_3^\infty = \frac{9}{2} = 4.5.$$

(d) Recall that $\sigma_X^2 = E[X^2] - (E[X])^2$. Now we calculated $E[X]$ in part (c), and we see that

$$E[X^2] = \int_\infty^\infty x^2 f(x)\,dx = \int_3^\infty x^2\frac{81}{x^4}\,dx = \int_3^\infty 81x^{-2} = -\frac{81}{x}^{-1}\Big|_3^\infty = 27.$$

So $\sigma_X = \sqrt{E[X^2] - (E[X])^2} = \sqrt{27 - 4.5^2} \sim 2.5981.$

[5] 4. Suppose that the joint probability distribution of X and Y is given by

$$f(x, y) = \frac{x + y}{30}, \quad \text{for } x = 0,\ 1,\ 2,\ 3,\ \ y = 0,\ 1,\ 2.$$

(a) Find the marginal distribution for X.
(b) Find the marginal distribution for Y.
(c) Find $P(X = 1, Y = 1)$, $P(X = 1)$, and $P(X = 1|Y = 1)$.
(d) Use the result from part (c) to determine whether X and Y are statistically independent. Explain your reasoning in at most one line.
(e) Find the covariance of X and Y.

Solution: Let's draw the table for the probabilities of the joint distribution, including the row and column totals, which give the marginal distributions:

		x				Row
		0	1	2	3	Totals
y	0	0	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{3}{30}$	$\frac{6}{30}$
	1	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{3}{30}$	$\frac{4}{30}$	$\frac{10}{30}$
	2	$\frac{2}{30}$	$\frac{3}{30}$	$\frac{4}{30}$	$\frac{5}{30}$	$\frac{14}{30}$
Column Totals		$\frac{3}{30}$	$\frac{6}{30}$	$\frac{9}{30}$	$\frac{12}{30}$	1

(a) Reading off the column totals in the above table and expressing as decimals, we see that the marginal distribution for X is given by the table:

x	0	1	2	3
$g(x)$	0.1	0.2	0.3	0.4

(b) Reading off the row totals gives the marginal distribution for Y:

y	0	1	2
$h(y)$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{7}{15}$

(c) $P(X = 1, Y = 1) = f(1,1) = 2/30 = 1/15$, reading off the table. The probability $P(X = 1)$ is the marginal probability $g(1) = 1/5$. Finally, the conditional probability is given by the formula:

$$P(X = 1|Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{1/15}{1/3} = 1/5.$$

[+1] (d) (bonus) In part (c) we had $P(X = 1|Y = 1) = P(X = 1) = 1/5$. This might suggest that X and Y are independent. However, we also have that

$$P(X = 1|Y = 2) = \frac{3/30}{14/30} = 3/14 \neq 1/5 = P(X = 1).$$

Any exception to the rule

$$P(X = x) = P(X = x|Y = y)$$

implies that X and Y are not statistically independent, so the random variables X and Y in this problem are **not** independent.

Note: Because of the bad wording, part (d) is treated as a bonus question on this assignment. If you gave the correct answer with proper reasoning, then you get an extra 1 point beyond 100%, and if you incorrectly concluded that X and Y were independent, then you lose no marks.

(e) Recall that $\sigma_{XY} = E[XY] - E[X]E[Y]$. The expectations $E[X]$ and $E[Y]$ can be computed from the marginal distributions:

$$E[X] = 0 * 0.1 + 1 * 0.2 + 2 * 0.3 + 3 * 0.4 = 2$$
$$E[Y] = 0 * \frac{1}{5} + 1 * \frac{1}{3} + 2 * \frac{7}{15} = \frac{19}{15},$$

and the expected value $E[XY]$ is a sum of 12 terms:

$$\begin{aligned} E[XY] = \frac{1}{30} * \big(&0 * 0 * 0 + 1 * 0 * 1 + 2 * 0 * 2 + 3 * 0 * 3 \\ &+ 0 * 1 * 1 + 1 * 1 * 2 + 2 * 1 * 3 + 3 * 1 * 4 \\ &+ 0 * 2 * 2 + 1 * 2 * 3 + 2 * 2 * 4 + 3 * 2 * 5\big) \\ =&\frac{1}{30}(72) = \frac{12}{5} \end{aligned}$$

So $\sigma_{XY} = E[XY] - E[X]E[Y] = \frac{12}{5} - (2)\left(\frac{19}{15}\right) = -\frac{2}{15}$.

[2] 5. Suppose that the engines on an airplane fail independently, each with probability 0.3. Assume that a plane makes a safe flight if at least half its engines do not fail.
(a) What is the probability of a successful flight of a 2-engine plane?
(b) What is the probability of a successful flight of a 4-engine plane?

Solution: Let X be the discrete random variable given by the number of motors that fail. Then X is given by a binomial distribution with $p = 0.3$. ("Success" here counts as a failed engine. We could instead define "success" as an engine that does not fail, in which case X would be the number of working engines and p would be $1 - 0.3 = 0.7$.)
(a) Here $n = 2$ and we want

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) = b(0; 2, 0.3) + b(1; 2, 0.3) \\ &= \binom{2}{0}(0.3)^0(0.7)^2 + \binom{2}{1}(0.3)^1(0.7)^1 = 0.49 + 2 * 0.21 = 0.91. \end{aligned}$$

(Alternatively, $P(X \leq 1) = 1 - P(X = 2) = 1 - \binom{2}{0}(0.3)^2(0.7)^0 = 0.91$.)
(b) Here $n = 4$ and we want

$$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) = b(0; 4, 0.3) + b(1; 4, 0.3) + b(2; 4, 0.3) \\ &= \binom{4}{0}(0.3)^0(0.7)^4 + \binom{4}{1}(0.3)^1(0.7)^3 + \binom{4}{2}(0.3)^2(0.7)^2 = 0.9163. \end{aligned}$$

[2] 6. Suppose that a student has a 25% probability of getting a perfect score on any assignment, and that the scores on separate assignments are independent.
(a) What is the probability that the second perfect score will occur on the sixth assignment?
(b) What is the expected number of assignments the student would write to get her first perfect score?

Solution: (a) Let X be the discrete random variable given by the number of assignments a student writes until the second perfect score. It has a negative binomial distribution with $k = 2$ (the number of perfect scores desired) and $p = 0.25$ (the probability of any one being perfect). We are asked for $P(X = 6)$; the formula on page 115 of the text gives:

$$P(X = 6) = b^*(6; 2, 0.25) = \binom{6-1}{2-1} 0.25^2 0.75^{6-2} \sim 0.09888$$

(b) Let Y be the discrete random variable given by the number of assignments the student writes to get her first perfect score. It is given by a geometric distribution with $p = 0.25$, so by Theorem 3.3, we have

$$E[Y] = \mu_Y = \frac{1}{p} = 4.$$

The expected number is 4.

[**+1**] 7. (bonus) A large company has an inspection system for the batches of small compressors purchased from vendors. A batch typically contains 15 compressors. In the inspection system, a random sample of 5 is selected and all are tested. Suppose there are 3 faulty compressors in the batch of 15. What is the probability that a given sample will have exactly 2 faulty compressors?

Solution: This is an example of the hypergeometric distribution, which we did not cover in class, but is found in section 3.3. The desired probability is

$$h(2; 15, 5, 3) = \frac{\binom{3}{2}\binom{15-3}{5-2}}{\binom{15}{5}} = \frac{20}{91} \sim 0.2198.$$

[/**25**]