

**Applied Ordinary Differential Equations**  
**ENGR 213 - Section F**  
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**Exam II (B)**

- (1) (6 points) Solve the homogeneous ODE

$$x^2y'' + 3xy' + 5y = 0.$$

**Solution:** The characteristic equation of this homogeneous Cauchy-Euler ODE is  $m(m-1) + 3m + 5 = 0$  or  $m^2 + 2m + 5 = 0$ . It has complex roots  $r = -1 \pm 2i$ . Hence the general solution of the ODE on  $x > 0$  is

$$y(x) = c_1x^{-1} \cos(2 \ln x) + c_2x^{-1} \sin(2 \ln x), \quad c_{1,2} = \text{constants.}$$

- (2) (14 points) Solve the initial value problem

$$y'' - 6y' + 9y = x, \quad y(0) = 0, \quad y'(0) = 1.$$

**Solution:** Consider first the associated homogeneous ODE:  $y'' - 6y' + 9y = 0$  with the characteristic equation  $r^2 - 6r + 9 = 0$  with  $r = 3$  as double root. Hence

$$y_c(x) = c_1e^{3x} + c_2xe^{3x}, \quad c_{1,2} = \text{constants.}$$

We now look for a particular solution  $y_p$  to the non-homogenous ODE. We'll use here the method of undetermined coefficients by setting  $y_p(x) = Ax + B$ . As  $y_p(x) = A$  and  $y_p''(x) = 0$ , we deduce that  $-6A + 9Ax + 9B = x \Rightarrow A = 1/9, -6A + 9B = 0$  thus  $B = 2A/3 = 2/27$  and  $y_p(x) = \frac{x}{9} + \frac{2}{27}$ .

Thus

$$y_{\text{general}}(x) = c_1e^{3x} + c_2xe^{3x} + \frac{x}{9} + \frac{2}{27}, \quad c_{1,2} = \text{constants.}$$

We'll now use the initial conditions to find  $c_{1,2}$ . As  $y(0) = 0$ , we have  $c_1 + 2/27 = 0 \Rightarrow c_1 = -2/27$ . Evaluating  $y'(x) = c_1(3e^{3x}) + c_2(e^{3x} + 3xe^{3x}) + 1/9$ , thus  $y'(0) = 3c_1 + c_2 + 1/9 = 1$ , implying  $c_2 = 10/9$ .

Therefore the solution of the IVP is

$$y(x) = -\frac{2}{27}e^{3x} + \frac{10}{9}xe^{3x} + \frac{x}{9} + \frac{2}{27}.$$

- (3) (10 points) Use the variation of parameters to solve the differential equation

$$y'' + y = \sin^2 x.$$

**Solution:** The complementary part of the solution follows from  $r^2 + 1 = 0 \Rightarrow r = \pm i$  and is

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

Considering  $y_1(x) = \cos x$ ,  $y_2(x) = \sin x$ , the Wronskian is  $W(x) = 1 \neq 0$  for all real  $x$ 's. To find the complementary solution we calculate  $W_1(x) = \det \begin{pmatrix} 0 & \sin x \\ \sin^2 x & \cos x \end{pmatrix} = -\sin^3 x$  and  $W_2(x) = \det \begin{pmatrix} \cos x & 0 \\ -\sin x & \sin^2 x \end{pmatrix} = \sin^2 x \cos x$ .

The method of variation of parameters gives  $y_p(x) = y_1(x)u_1(x) + y_2(x)u_2(x)$ , where  $u_1'(x) = W_1(x)/W(x)$  and  $u_2'(x) = W_2(x)/W(x)$ .

Integrating (by taking  $u = \sin x$   $du = \cos x dx$ ) and taking the constant of integration to be zero, we have

$$u_2(x) = \int (\sin^2 x \cos x) dx = \int u^2 du = \frac{u^3}{3} = \frac{\sin^3 x}{3}.$$

On the other hand,

$$u_1(x) = - \int \sin^3 x dx = - \int \sin x (1 - \cos^2 x) dx = \int (1 - u^2) du = u - \frac{u^3}{3} = \cos x - \frac{\cos^3 x}{3},$$

where above we used the fundamental identity of trigonometry ( $\sin^2 x + \cos^2 x = 1$ ) and the substitution  $u = \cos x$ ,  $du = -\sin x dx$ .

Consequently,

$$y_p(x) = \cos x \cdot \left( \cos x - \frac{\cos^3 x}{3} \right) + \sin x \cdot \frac{\sin^3 x}{3}$$

and

$$y(x) = c_1 \cos x + c_2 \sin x - \frac{\cos^4 x}{3} + \cos^2 x + \frac{\sin^4 x}{3}, \quad c_{1,2} = \text{arbitrary constants.}$$

- (4) (10 points) A mass weighing 64 pounds stretches a spring 0.32 foot. Determine the equation of motion if the mass is initially released from a point 6 inches above the equilibrium position with a downward velocity of 5 ft/s. What is the instantaneous velocity at the first time when the mass passes through the equilibrium position?

**Solution:** The equation of motion is  $mx'' + kx = 0$ , where  $m = 64/32$  slug and  $k = 64/0.32 = 200$  ft/lb. Thus

$$x'' + 100x = 0 \Rightarrow x(t) = c_1 \cos 10t + c_2 \sin 10t, \quad c_{1,2} = \text{constants.}$$

To determine the constants, use the initial conditions. As  $x(0) = -1/2$  ft,  $c_1 = -1/2$ . Additionally,  $x'(t) = -10c_1 \sin 10t + 10c_2 \cos 10t$ , thus  $x'(0) = 5$  ft/sec and  $10c_2 = 5$ .

Consequently,  $x(t) = -\frac{1}{2} \cos 10t + \frac{1}{2} \sin 10t$  and  $x'(t) = 5 \sin 10t + 5 \cos 10t$ . To find the time when the mass passes through the equilibrium position, set  $x(t) = 0$ . Note that this implies  $\tan(10t) = 1$  whose first positive solution is for  $10t = \pi/4$ . So

$$x'(\pi/40) = 5 \frac{\sqrt{2}}{2} + 5 \frac{\sqrt{2}}{2} = 5\sqrt{2} \text{ ft/sec}$$

is the answer needed.