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Faculté de génie

Département de
Génie Mécanique



uOttawa

L'Université canadienne
Canada's university

University of Ottawa
Faculty of Engineering

Department of
Mechanical Engineering

MCG 4303
Mechanical Vibration Analysis

FINAL EXAMINATION

Length of Examination: 3 hours

**April 19, 2006
9:30am-12:30pm**

Professor: Natalie Baddour

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Family Name: _____

Other Names: _____

Student Number: _____

Signature _____

Closed book.

If you do not understand a question, clearly state an assumption and proceed.
There are **SIX** questions. Attempt all questions.

No calculators are allowed.

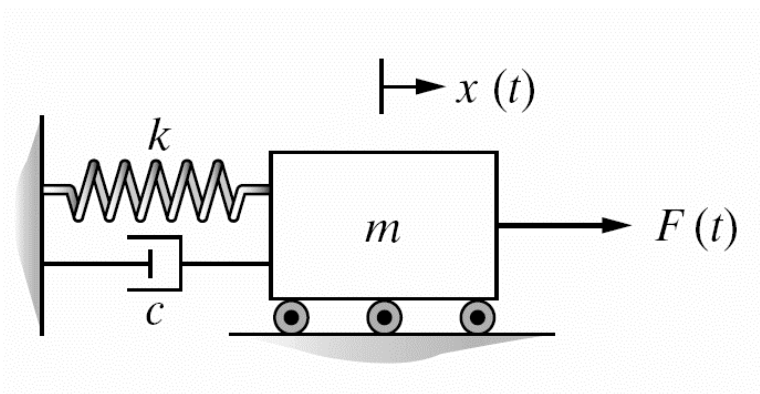
At the end of the exam, when time is up:

- Stop working and turn your exam upside down.
- Remain silent.
- Do not move or speak until all exams have been picked up, and a TA or the Professor gives the go-ahead to leave.

Question #1 (16 marks)

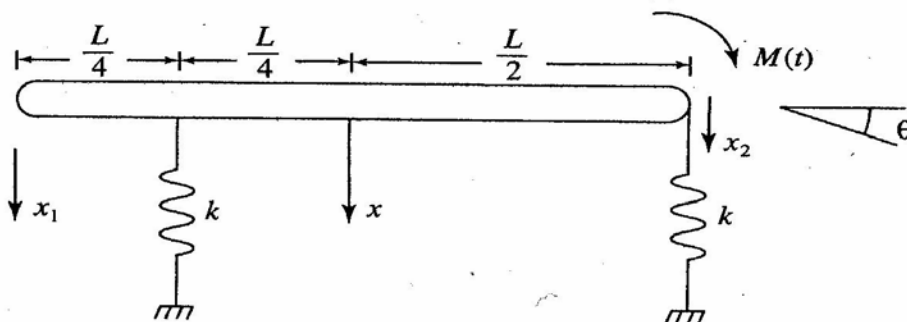
Consider the 1 degree-of-freedom mass-spring-damper system shown in the figure. The mass m is 1kg, the stiffness of the spring k is 2 N/m and the damping constant c is 2 Ns/m.

- Write down the equation of motion of this system.
- Find the general **free vibration** response of the system. Is it under-damped, critically damped or overdamped?
- Find the natural frequency, damping coefficient and damped frequency of this system.
- If $F(t)$ is given by $F(t) = \cos(t)$, find the **steady-state response** to this harmonic forcing.
- How is the total response related to the transient and steady-state responses?

**Question #2** (16 marks)

Consider the system shown in the figure. Use x and θ as generalized coordinates.

- Use Lagrange's equations to derive the equations of motion for x and θ .
- Write the equations of motion in matrix form.
- Identify the mass and stiffness matrices of this system.



Question #3 (16 marks)

A 2 degree-of-freedom vibrating system is described by the following equations of motion, where $\delta(t)$ is the unit impulse function.

$$\begin{bmatrix} m & 0 \\ 0 & 4m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & 16k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \delta(t) \\ 4 \end{bmatrix}.$$

Find the total response given zero initial conditions: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Question #4 (16 marks)

A vibrating system is described by the following equations of motion:

$$\begin{bmatrix} m & 0 \\ 0 & 3m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 4k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Find the natural frequencies and modeshapes.
- Find the resulting motion if the initial conditions are given as

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Question #5 (16 marks)

The equations of motion of a vibrating system are given by

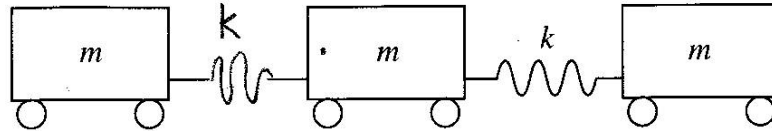
$$\begin{bmatrix} 3m & \frac{1}{2}mL \\ \frac{1}{2}mL & J + \frac{1}{4}mL^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & \frac{1}{2}kL^2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- What kind of coupling does this system have?
- Use Raleigh's method to find an estimate of the fundamental frequency.
- Use Dunkerley's method to find an estimate of the fundamental frequency.
- Which estimate is smaller?

Question #6 (20 marks)

The mass matrix of the vibrating system shown in the figure is given by

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and the modal vectors are given by } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$



- Find the modal matrix.
- Normalize the modal matrix with respect to the mass matrix to find the $[M]$ -normalized modal matrix.
- Find the natural frequencies of the system given that the stiffness matrix is

$$K = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

- What is the physical interpretation of the zero frequency?

Useful formulas

Quadratic formula: roots of ax^2+bx+c are given by $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\int u dv = uv - \int v du$$

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$\sin(u) = \frac{e^{iu} - e^{-iu}}{2i} = \text{Im}\{e^{iu}\} \quad \cos(u) = \frac{e^{iu} + e^{-iu}}{2} = \text{Re}\{e^{iu}\}$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t) dt = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} x(t) dt$$

$$a_n = \frac{2}{\tau} \int_0^{\tau} x(t) \cos\left(\frac{2n\pi t}{\tau}\right) dt = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} x(t) \cos(n\omega t) dt$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} x(t) \sin\left(\frac{2n\pi t}{\tau}\right) dt = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} x(t) \sin(n\omega t) dt$$

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i$$

$$\text{Convolution: } f * g = \int_0^t f(\tau) g(t - \tau) d\tau$$

Short Table of Laplace Transforms

$f(t)$	$F(s) = L[f(t)]$
1	$\frac{1}{s}$
$\delta(t)$	1
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$u(t-a)$	$\frac{1}{s} e^{-as}$

Trigonometric Identities

Reciprocal

$$\begin{aligned} \sec x &= \frac{1}{\cos x} & \csc x &= \frac{1}{\sin x} \\ \tan x &= \frac{\sin x}{\cos x} & \cot x &= \frac{\cos x}{\sin x} \\ \cot x &= \frac{1}{\tan x} & \tan x &= \frac{1}{\cot x} \end{aligned}$$

Pythagorean

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ 1 + \tan^2 x &= \sec^2 x \\ 1 + \cot^2 x &= \csc^2 x \end{aligned}$$

Negative Angle

$$\begin{aligned} \sin(-x) &= -\sin x \\ \cos(-x) &= \cos x \\ \tan(-x) &= -\tan x \end{aligned}$$

Addition and Subtraction

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Periodicity

$$\sin(x + 2\pi) = \sin x \quad \cos(x + 2\pi) = \cos x$$

$$\csc(x + 2\pi) = \csc x \quad \sec(x + 2\pi) = \sec x$$

$$\tan(x + \pi) = \tan x \quad \cot(x + \pi) = \cot x$$

Cofunction

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \quad \cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\tan x = \cot\left(\frac{\pi}{2} - x\right) \quad \cot x = \tan\left(\frac{\pi}{2} - x\right)$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \quad \csc x = \sec\left(\frac{\pi}{2} - x\right)$$

Product

$$\sin x \cos y = \frac{1}{2} (\sin(x + y) + \sin(x - y))$$

$$\sin x \sin y = \frac{1}{2} (\cos(x - y) - \cos(x + y))$$

$$\cos x \cos y = \frac{1}{2} (\cos(x + y) + \cos(x - y))$$

$$\cos x \sin y = \frac{1}{2} (\sin(x + y) - \sin(x - y))$$

Factoring

$$\sin x + \sin y = 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$

$$\sin x - \sin y = 2 \cos\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$

Double Angle

$$\sin(2x) = 2 \sin x \cos x \quad \cos(2x) = 1 - 2 \sin^2 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x \quad \tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\cos(2x) = 2 \cos^2 x - 1$$

Half-Angle

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}} \quad \tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}} \quad \tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$$