DGD (circle one):
DGD1 (in DMS1120)
DGD2 (in STEB0138)

Last name: Solutions
First name:
Student number:

Marks: /11

MAT 1348B — Eighth Homework Assignment (Prof. P. Scott) — Due March 25, 2015 by 3:30pm

Instructions: Print this document to hand in. Note your DGD. Show all relevant work to receive full credit. *You may work with friends, but you must write up your assignment yourself, in your own words, in your own notation!* You may write on both sides of the paper or insert additional pages if necessary. Please staple the pages. Submit the assignment to your TA in the DGD or in the appropriate submission box in the Department of Mathematics and Statistics. Late assignments will not be accepted. **NOTE:** I will put some notes on modular arithmetic on Blackboard.

1. At random, you choose a subset S of $\mathbb{Z} \times \mathbb{Z}$. What is the smallest cardinality of S that guarantees that your set S contains two elements (m, n) and (m', n') such that $m \equiv m'$ (mod 6) and $n \equiv n'$ (mod 9)? We write congruence $m \underset{6}{\equiv} m'$ and $n \underset{9}{\equiv} n'$. [3pts]

Answer: Consider $\mathbb{Z}_6 \times \mathbb{Z}_9$. This consists of points (x, y), where $x \in \mathbb{Z}_6 = \{0, .., 5\}$, $y \in \mathbb{Z}_9 = \{0, .., 8\}$ are the remainders mod 6 and mod 9, respectively. So consider these 54 grid points $\{(x,y) \mid x \in \{0,..,5\}, y \in \{0,..,8\}\}$. These can be used as labels for boxes. We distribute points $(m,n) \in S$ into boxes by saying: $(m,n) \in$ box labelled (x,y) iff $m \equiv x$ mod 6 and $n \equiv y$ mod 9.

Suppose we have $54+1 = 55$ points in S. Then by the Pigeonhole Principle, 2 points live in the same box. Say $(m,n), (m',n') \in S$ lie in the same box labelled (x,y). Then $m \underset{6}{\equiv} x$ and $m' \underset{6}{\equiv} x$ and $n \underset{9}{\equiv} y$ and $n' \underset{9}{\equiv} y$. By commutativity and transitivity of $\underset{n}{\equiv}$ (for $n = 6, 9$) we get $m \underset{6}{\equiv} m'$ and $n \underset{9}{\equiv} n'$, as required. $\therefore |S| = 55$ suffices.

2. How many integers in the set $\{100, 101, \ldots, 1000\}$ are divisible by 4 or 7? (This is **inclusive or**.) How many are divisible by neither? [4pts]

Let $A = \{n \in \mathbb{Z}^+ \mid 100 \le n \le 1000,\ 4 \mid n\}$
$B = \{n \in \mathbb{Z}^+ \mid 100 \le n \le 1000,\ 7 \mid n\}$

We want $|A \cup B|$ and $|\overline{A \cup B}|$.
We know (by Inclusion-Exclusion)
$|A \cup B| = |A| + |B| - |A \cap B|$.

Now $A = \{n \in \mathbb{Z}^+ \mid 100 \le n \le 1000 \wedge n = 4k, \text{ some } k\}$
$= \{n \in \mathbb{Z}^+ \mid 100 \le 4k \le 1000 \wedge n = 4k\}$
$\therefore \frac{100}{4} \le k \le \frac{1000}{4}$ and $k \in \mathbb{Z}^+$. $\therefore 25 \le k \le 250$
$\therefore |A| = 250 - 25 + 1 = \boxed{226.}$

Similarly, $B = \{n \in \mathbb{Z}^+ \mid 100 \le 7k \le 1000,\ n = 7k\}$
$\therefore \frac{100}{7} \le k \le \frac{1000}{7}$ $\therefore 15 \le k \le 142$ $\therefore |B| = 142 - 15 + 1 = \boxed{128}$

$A \cap B = \{n \in \mathbb{Z}^+ \mid 100 \le n \le 1000,\ 28 \mid n\} = \{n \mid 100 \le 28k \le 1000,\ n = 28k\}$
$\therefore \frac{100}{28} \le k \le \frac{1000}{28}$, $k \in \mathbb{Z}^+$ $\therefore 4 \le k \le 35$. $\therefore |A \cap B| = 35 - 4 + 1 = \underline{32}$.

$\therefore |A \cup B| = 226 + 128 - 32 = \boxed{322}$

Let $\mathcal{U} = \{100, 101, \ldots, 1000\}$. $|\mathcal{U}| = 1000 - 100 + 1 = 901$.
$|\overline{A \cup B}| = |\mathcal{U}| - |A \cup B| = 901 - 322 = \boxed{579}$.

<u>Answers</u> $|A \cup B| = 322$
$|\overline{A \cup B}| = 579$.

3. A student wishes to arrange on a shelf, in a row, 5 out of her 12 textbooks (which includes a copy of Rosen's Discrete Mathematics and a copy of Stewart's Calculus). How many ways can this be done if [4pts]

   (a) at least one of Rosen and Stewart must be on the shelf?

   (b) exactly one of Rosen and Stewart must be on the shelf?

Ans

(a) # of ways of arranging 5 out of 12 : $P(12,5)$.

ways of arranging 5 out of all books except Rosen, Stewart = # ways of arranging 5 out of 10 books : $P(10,5)$

$\therefore$ # ways of arranging 12 books so that at least one of R or S is on the shelf is $\boxed{P(12,5) - P(10,5)}$

$= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 - 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 64,800.$

(b). To arrange 5 of 12 books including one of R. or S.

(i) # ways to choose 4 out of 10 $= \binom{10}{4}$ (this excludes R., S.).

(ii) # ways to choose one of R, S : $\binom{2}{1}$.

(iii) # permutations of chosen 5 : $5!$

Ans: $\binom{10}{4} \cdot \binom{2}{1} \cdot 5! = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 2$

$= 50,400.$