The University of British Columbia

Midterm Examination - October 14, 2011

MATH 104, Section 105

Closed book examination Time: 50 minutes

Last Name Thompson First William

Signature ~~~

Student Number ?

Special Instructions:

No memory aids are allowed. ~~One basic scientific calculator, WITH COVER REMOVED, may be used~~. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		7
2		10
3		7
4		7
5		9
6		5
Total		45

[7] 1.

(a) Carefully state the definition of the derivative of a function $f(x)$ at a point $x = a$.

The derivative of $f(x)$ at the point $x=a$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \left(\text{or } \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right)$$

provided the limit exists.

(b) Suppose $f(x) = \dfrac{3}{x+2}$. Show that $f'(a) = \dfrac{-3}{(a+2)^2}$ using the definition of derivative. NO credit will be given for any other method.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{3}{(a+h+2)} - \frac{3}{a+2}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{3a+6-3(a+h)-6}{(a+h+2)(a+2)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-3h}{(a+h+2)(a+2)}\right)$$

$$= \lim_{h\to 0} \frac{-3}{(a+h+2)(a+2)}$$

$$\boxed{f'(a) = \frac{-3}{(a+2)^2}}$$

[10] 2. The video game publisher Tornado Entertainment are the creators of the Starcruft video game. They have found that when they sell Starcruft at the price of \$40 per copy, they sell 36,000 units per month. For each additional dollar the price is increased, they sell 600 fewer units per month. The fixed costs of production per month are \$100,000, and each copy of the game costs Tornado \$15 to make.

(a) Find the linear demand equation for the Starcruft video game. Use p for the unit price and q for the monthly demand.

From the information given, if $\begin{cases} p = 40, & q = 36000 \\ p = 41, & q = 36000 - 600 \end{cases}$

if $q = mp + b$

$$m = \frac{(36000 - 600) - 36000}{41 - 40} = -600$$

$q = -600p + b$

$36000 = -600(40) + b$

$36000 + 24000 = b$

$b = 60000$

$\therefore q = -600p + 60{,}000$ (2)

(b) Find the monthly cost function $C(q)$ *as a function of* q.

$C(q) = 100{,}000 + 15q$ (1)

fixed cost; variable cost.

(c) Find the the monthly revenue function $R(q)$ *as a function of* q.

$R = p \cdot q$

What is $p(q)$? $\rightarrow q = -600p + 60000$

$q - 60000 = -600p$

$\left(\frac{-1}{600}q + 100 = p\right)$

$\therefore R(q) = \left(\frac{-1}{600}q + 100\right)q = \frac{-1}{600}q^2 + 100q$ (2)

(d) Find the monthly profit function $P(q)$ *as a function of* q.

$$P = R - C$$

$$P(q) = \frac{-1}{600}q^2 + 100q - (100{,}000 + 15q)$$

$$\boxed{P(q) = \frac{-1}{600}q^2 + 85q - 100{,}000}$$ (2)

(e) Find the marginal profit function $MP(q)$. If the price is currently \$50 per copy, will profits increase or decrease if we raise this price by a small amount? (3)

$$MP(q) = \frac{dP(q)}{dq}$$

$$\boxed{MP(q) = \frac{-1}{300}q + 85}$$ (1)

If $p = 50$, then $q = -600(50) + 60{,}000$

$= -30000 + 60000$

$= 30{,}000$

$\Rightarrow$ At $p = 50$, $MP(30{,}000) = -15$

$$\frac{dP}{dp} = \underbrace{\frac{dP}{dq}\cdot\frac{dq}{dp}} = -15 \cdot (-600) = 9000 > 0$$ (2)

chain rule

Profits increase if the price is raised

[7] **3.** Consider the function

$$f(x) = \begin{cases} ax^2 + x \text{ if } x \leq 1 \\ b - 2 \text{ if } x > 1. \end{cases}$$

(a) Find the values of the parameters a and b such that $f(x)$ is continuous at $x = 1$.

3 To have continuity we must have that

1) $\lim_{x\to 1} f(x)$ exists
2) $f(1)$ exists
3) $\lim_{x\to 1} f(x) = f(1)$

Clearly 2) is satisfied and 3) is satisfied as long as 1) is satisfied

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} b-2 = b-2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} ax^2+x = a+1$

$\therefore$ for 1) to be satisfied we need $a+1 = b-2$

$\Rightarrow$ For f to be continuous at $x=1$, we need $b = a+3$, $a \in \mathbb{R}$

(b) For which values of a and b that you found in part (a) is $f(x)$ differentiable at $x = 1$?

4 For differentiably we need $f'(1)$ to exist:

ie) $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ must exist

using the result from a)

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{b-2-(a+1)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{a(1+h)^2+(1+h)-(a+1)}{h} = \lim_{h\to 0} \frac{a+2ah+h^2+1+h-a-1}{h}$$

$$= \lim_{h\to 0} 2a+1+h = 2a+1$$

$\therefore$ $f'(1)$ exists if $2a+1=0 \Rightarrow \boxed{a = \frac{-1}{2}} \Rightarrow \boxed{b = \frac{5}{2}}$

[7] 4. Find the equation of the tangent line to the graph $y = x^2e^x$ at the points:

4 (i) $(1, e)$,

3 (ii) $(2, 4e^2)$.

$y = x^2e^x \Rightarrow \frac{dy}{dx} = 2xe^x + x^2e^x = e^x(2x + x^2)$ ✓

(by product rule) ✓

Recall the tangent line for a function $y = f(x)$ at a point $x = a$ is

$y = f'(a)(x - a) + f(a)$ ✓

$\Rightarrow$ i) $(a, f(a)) = (1, e)$, $f'(1) = \left.\frac{dy}{dx}\right|_{x=1} = 3e$ ✓

$\therefore y = 3e(x - 1) + e \Rightarrow \boxed{y = 3e \cdot x - 2e}$ ✓

ii) $(a, f(a)) = (2, 4e^2)$, $f'(2) = \left.\frac{dy}{dx}\right|_{x=2} = 8e^2$ ✓

$\therefore y = 8e^2(x - 2) + 4e^2 \Rightarrow \boxed{y = 8e^2 \cdot x - 12e^2}$ ✓

[9] 5.

(a) Evaluate the limit $\lim_{x\to 3} \dfrac{x^2-4x+3}{x^2-2x-3}$.

3

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^2-2x-3} = \lim_{x\to 3} \frac{(x-3)(x-1)}{(x-3)(x+1)} = \lim_{x\to 3} \frac{(x-1)}{(x+1)} \cdots$$

$$\cdots \longrightarrow = \frac{2}{4} = \boxed{\frac{1}{2}}$$

(b) Find $f'(x)$ given that $f(x) = \dfrac{e^x+x^2}{12x^2+x}$. (Do not simplify your answer).

3

$$\boxed{f'(x) = \frac{(12x^2+x)(e^x+2x)-(24x+1)(e^x+x^2)}{(12x^2+x)^2}}$$

by the quotient rule

(c) Find $f'(x)$ given that $f(x) = \sin(e^{\sqrt{x}})$.

3

Using chain rule

$$f'(x) = \cos(e^{\sqrt{x}}) \cdot \frac{d}{dx}(e^{\sqrt{x}})$$

$$= \cos(e^{\sqrt{x}}) \cdot e^{\sqrt{x}} \cdot \frac{d}{dx}\sqrt{x} \qquad (\sqrt{x} = x^{1/2})$$

$$= \cos(e^{\sqrt{x}}) \cdot e^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2}$$

$$\boxed{f'(x) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}\cos(e^{\sqrt{x}})}$$

[5] **6.** Use the Intermediate Value Theorem to show that the equation $(3^x - 2)(x^2 + 1) = 1$ has a solution.

Clearly the curve $(3^x-2)(x^2+1)$ is continuous, ✓ because it is a product of two continuous functions:

Taking $x = 0 \Rightarrow (3^0-2)(1) = -1 < 1$

$x = 1 \Rightarrow (3^1-2)(1^2+1) = 2 > 1$

∴ By the intermediate value theorem, the exists a number $c \in (0,1)$ such that

$$(3^c-2)(c^2+1) = 1$$

The End