Solutions : Hofstra & Scott versions.

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Discrete Mathematics for Computing MAT1348B

Practice Exam

5 April 2015
Prof. Philip Scott

Instructions:

- This is a three-hour *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 21 questions on 17 pages. Page 17 provides additional work space. Do not detach it.
- Questions 1-5 are multiple-choice. You must enter the letter corresponding to each correct answer in the table preceding Question 1. No partial marks will be given for other work.
- Questions 6-9 are true/false. You must circle the correct response. You need not justify your answers.
- Questions 10-17 are short-answer. Write the final answer in the appropriate answer box, and briefly justify your answer where required.
- Questions 18-21 are long-answer. You must clearly show all relevant steps in your solution to receive full marks. Clearly indicate the final answer.
- Be sure to read carefully and follow the instructions for the individual problems.
- For rough work, you may use the back pages. Do not use scrap paper of your own.
- Use proper mathematical notation and terminology.
- If you require clarification, raise your hand.
- Good luck!

Seat number:________

Last name:______________________________

First name:______________________________

Student number:__________________________

Signature:______________________________

Questions 1–5 are multiple choice. Enter the **letter** corresponding to each correct answer in the appropriate box below.

Question	1	2	3	4	5
Answer					

1. How many binary strings of length 10 start with 01 and contain at most four 0s?

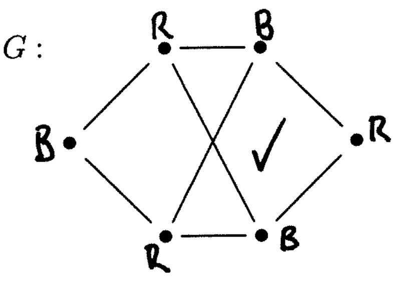
A. 12 **B.** 176 [✓ **C.** 93] **D.** 255 **E.** 32 **F.** 194
G. None of the above.

String looks like $01\underbrace{--------}_{\text{at most 3 0s}}$

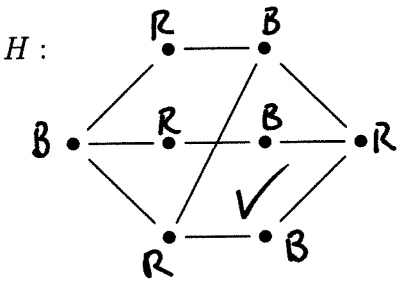
Exactly 3 0s : $\binom{8}{3} = 56$
" 2 0s : $\binom{8}{2} = 28$
" 1 0 : $\binom{8}{1} = 8$
" 0 0s : $\binom{8}{0} = 1$
At most 3 : $+$ $\underline{93}$

[2pts] 2. Which of the following graphs are bipartite?

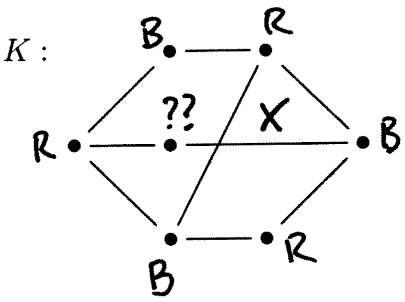
G:



H:



K:



A. Only G **B.** Only H **C.** Only K **D.** Only G and K
[✓ **E.** Only G and H] **F.** All of them. **G.** None of them.

3. Let $A = \{0,1,2,3,4\}$ and $B = \{0,2,4,6,8\}$. The number of surjective functions from A to B is:

A. 1 **B.** 5 [✓ **C.** 5!] **D.** 2^5 **E.** 5^5 **F.** 5^2
G. None of the above.

Note $|A| = |B| = 5$, so any surjective function is bijective. Hence the number is $5!$

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4. Let A, B, C be arbitrary subsets of the universal set $\mathcal{U}$. Which of the following statements is necessarily true?

A. $A \cap B \subseteq A \cap C$ implies $B \subseteq C$
B. $A \cup B \cup C = \mathcal{U}$ implies $(A \cup B) - (A \cup C) = \emptyset$
[✓ **C.**] $A \subseteq B \cup C$ implies $A - B \subseteq C - B$.
D. $A \cap B \subseteq C$ implies $(C - A) \cap (C - B) = \emptyset$.
E. None of the above.

A. is false, let $A = \emptyset$, $B = \{1\}$, $C = \{2\}$

B. is false, let $B = \mathcal{U}$, $A = C = \emptyset$.

C. is true; if $A \subseteq B \cup C$ and $x \in A - B$, then $x \in A$, $x \notin B$ so $x \in C$.

D. is false; let $A = B = \emptyset$, $C \neq \emptyset$.

5. From an urn containing balls numbered 1-10, we randomly draw three different balls, and record the sum of the three numbers. What is the smallest number of times we need to repeat this procedure to guarantee that the same sum shows at least twice?

A. 22 [✓ **B.** 23] **C.** 27 **D.** 28 **E.** 120 **F.** 121
G. None of the above.

The smallest possible sum is $1+2+3 = 6$
" largest " " " $8+9+10 = 27$.

Thus there are 22 possible outcomes; by the PHP we need at least 23 trials.

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6. Which of the following is logically equivalent to $(\neg b \wedge a) \vee (\neg a \wedge c) \vee (\neg b \wedge c)$?

A. $\neg(a \wedge c) \leftrightarrow (c \vee b)$ **B.** $\neg(a \wedge \neg b) \rightarrow (\neg a \wedge \neg c)$ **C.** $(a \wedge b) \leftrightarrow (c \vee \neg a)$
[✓ **D.**] $\neg(a \wedge b) \leftrightarrow (a \vee c)$ **E.** $\neg(c \wedge a) \rightarrow (a \wedge b)$ **F.** $\neg(a \wedge c) \leftrightarrow (c \vee b)$

$$\neg(a\wedge b) \Leftrightarrow (a\vee c) \equiv \big(\neg(a\wedge b)\wedge(a\vee c)\big) \vee \big((a\wedge b)\wedge\neg(a\vee c)\big)$$

$$\equiv \big((\neg a\vee\neg b)\wedge(a\vee c)\big) \vee \underbrace{\big((a\wedge b)\wedge\neg a\wedge\neg c\big)}_{\bot}$$

$$\equiv \underbrace{(\neg a\wedge a)}_{\bot}\vee(\neg a\wedge c)\vee(\neg b\wedge a)\vee(\neg b\wedge c)$$

(Where $\bot$ = <u>False</u>)

True/false questions — circle T (true) or F (false). You need not justify your answers.

7. For each of the statements below, determine whether it is true or false. *Circle each correct answer.*

1) For any set A, $A \in \mathcal{P}(A)$ $A \subseteq A$. $\therefore A \in P(A)$. [✓ T] F

2) For all sets A and B, $\mathcal{P}(B) \subseteq \mathcal{P}(A)$ implies $B \subseteq A$ [✓ T] F

3) For all sets A and B, $A \cup B \in \mathcal{P}(B)$ implies $A \subseteq B$ [✓ T] F

4) $\{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\emptyset)$ T [✓ F]

5) The relation $\{(n, n+k) | n \in \mathbb{N}, k \in \mathbb{N}^+\}$ is transitive [✓ T] F

6) There exists a relation on $\mathbb{Z}$ that is both symmetric and anti-symmetric. [✓ T] F

Ans: (2): $B \in P(B)$, since $B \subseteq B$. $\therefore B \in P(A)$. $\therefore B \subseteq A$. ✓

(3) To show $A \subseteq B$, suppose $x \in A$. Must prove $x \in B$. But if $x \in A$, then $x \in A \vee x \in B$. $\therefore x \in A \cup B$. But $A \cup B \in P(B)$. $\therefore A \cup B \subseteq B$. $\therefore x \in B$. Hence $A \subseteq B$. ✓

(4) $P(\emptyset) = \{S \mid S \subseteq \emptyset\}$. Since $\emptyset \subseteq \emptyset$, we see $\emptyset \in P(\emptyset)$. In fact, $P(\emptyset) = \{\emptyset\}$. But notice $\{\emptyset\} \notin P(\emptyset)$, since if $\{\emptyset\} \in P(\emptyset)$, then that would mean $\{\emptyset\} \subseteq \emptyset$, which is obviously false — since if $\{\emptyset\} \subseteq \emptyset$ then $\{\emptyset\} = \emptyset$ (Why? More generally, if $A \subseteq \emptyset$ then $A = \emptyset$. But clearly $\{\emptyset\} \neq \emptyset$, since $\{\emptyset\}$ has cardinality $\underline{1}$!)

(5) Call the relation R. if $(n,m) \in R \wedge (m,p) \in R$ then $m = n+k$ and $p = m+l$. $\therefore p = m+l = (n+k)+l = n+(k+l)$. Also $k+l \in \mathbb{Z}^+ = \mathbb{N}^+$.

$\therefore (n,p) \in R$. $\therefore (n,m) \in R \wedge (m,p) \in R \longrightarrow (n,p) \in R$.

So R is transitive.

(6) Take $R = \{(x,x) \mid x \in \mathbb{Z}\}$

8. Consider the following function:

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^+; \quad f(x,y) = (xy + x, 2^y)$$

Which of the following statements about f are true? *Circle each correct answer below.*

f is injective. T [✓ F]

f is surjective. T [✓ F]

f is bijective. T [✓ F]

f is not injective, e.g. $f(0,-1) = f(1,-1)$

f is not surjective: $(1, \frac{1}{2})$ is not of the form $f(x,y)$,

since $f(x,y) = (1, \frac{1}{2}) \Rightarrow 2^y = \frac{1}{2} \Rightarrow y = -1$,

but then $xy + x = 0$.

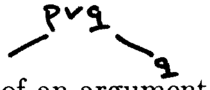
f is not bijective as it is not injective / surjective.

9. For each of the statements below, determine whether it is true or false. *Circle each correct answer below.*

Every proposition is logically equivalent to a proposition containing only the connectives $\neg, \to$. [✓ T] F

Yes, bring to DNF, replace $\wedge$'s by $\vee$'s using de Morgan, and use $p \vee q \equiv \neg p \to q$.

If the complete truth-tree of a propositional formula P has only open branches, then P is a tautology. T [✓ F]



not a tautology

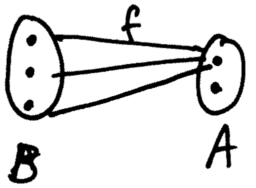
If the set of premises of an argument is inconsistent, then the argument is valid. [✓ T] F

This makes $p_1 \wedge \cdots \wedge p_k \to q$ vacuously true.

For any two functions $f, g : \mathbb{N} \to \mathbb{N}$, if f and g are bijective then so is $g \circ f$. [✓ T] F

The inverse is $f^{-1} \circ g^{-1}$.

For any two finite sets A, B, if $|A| < |B|$, then every function $f : B \to A$ is surjective but not injective. T [✓ F]



not surjective.

Short-answer questions — write your final answer in the answer box. Wherever indicated, you must briefly justify your answers to receive full marks.

10. In the following question, you do not have to justify your answers.

(a) Give the definition of a *transitive* relation.

Answer:

See back

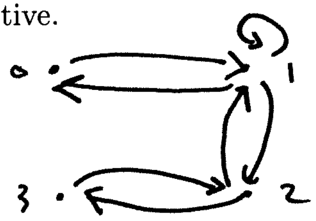
(b) Give the definition of a *symmetric* relation.

Answer:

See back

(c) Give an example of a relation on the set $A = \{0, 1, 2, 3\}$ which is symmetric but not transitive.

Answer:



$0 R 1$, $1 R 2$ but not $0 R 2$

(d) Give an example of a relation on the set $B = \{0, 1\}$ which is symmetric but not transitive.

Answer:

$0 R 1$ and $1 R 0$
but not $0 R 0$

11. From a group of 12 men and 15 women, a committee consisting of 6 people is chosen. In how many ways is this possible if the committee must contain at least one, but no more than four women? *Your answer may include unevaluated factorials, binomial coefficients, powers, products, or sums.*

Answer: $\binom{15}{1}\binom{12}{5} + \binom{15}{2}\binom{12}{4} + \binom{15}{3}\binom{12}{3} + \binom{15}{4}\binom{12}{2}$

Justification:

Exactly 1 woman : $\binom{15}{1}\cdot\binom{12}{5}$

" 2 women : $\binom{15}{2}\cdot\binom{12}{4}$

" 3 " $\binom{15}{3}\cdot\binom{12}{3}$

" 4 " $\binom{15}{4}\cdot\binom{12}{2}$

Between 1 and 4 : ______ +

12. Find a proposition in DNF equivalent to $(a \to b) \leftrightarrow (c \wedge \neg a)$.

Answer: $(\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge \neg b \wedge c)$

Justification:

	a	b	c	$(a \to b)$	$\leftrightarrow$	$(c \wedge \neg a)$
	0	0	0	1	0	0
→	0	0	1	1	1	1
	0	1	0	1	0	0
→	0	1	1	1	1	1
→	1	0	0	0	1	0
→	1	0	1	0	1	0
	1	1	0	1	0	0
	1	1	1	1	0	0

Or you could use a truth-tree.

13. How many integers between 200 and 1000 (inclusive) are divisible by 8 or by 12?

Answer: 135

Justification: Let A_i = # integers in [200,1000] div. by i.

$A_8 = \{k \mid 200 \le 8k \le 1000\} = \{k \mid 25 \le k \le 125\}$. $|A_8| = 125-25+1 = 101$

$A_{12} = \{k \mid 200 \le 12k \le 1000\} = \{\underset{\in \mathbb{Z}^+}{k} \mid \frac{200}{12} \le k \le \frac{1000}{12}\} =$

$= \{k \mid 16\tfrac{2}{3} \le k \le 83\tfrac{1}{3}\} = \{k \in \mathbb{Z}^+ \mid 17 \le k \le 83\}$ $\therefore |A_{12}| = 83-17+1 = 67$

$A_{24} = \{k \mid 200 \le 24k \le 1000\} = \{k \mid 8\tfrac{1}{3} \le k \le 41\tfrac{2}{3}\}$

$= \{k \mid 9 \le k \le 41\}$. $|A_{24}| = 41-9+1 = 33$

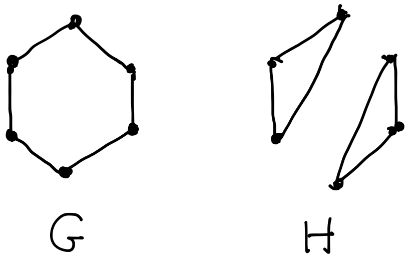
$\underline{\text{Inclusion - Exclusion}}$ $|A_8| + |A_{12}| - |A_{24}|$

$= 101 + 67 - 33 = \boxed{135}$

14. Give an example of simple graphs G and H such that all of the following conditions are met:

- G and H are not isomorphic
- G and H both have 6 vertices
- G and H both have 6 edges
- G and H have the same degree sequence

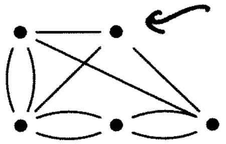
Answer:



Not isomorphic since one of them (H) is not connected.

(No justification is needed.)

15. Does the following graph admit an Euler tour?



has degree 3

has degree 5

Answer: YES [✓ NO]

Justification: For an Euler tour to be possible, all vertices have to have even degree.

16. Determine the coefficient of x^{11} in the expansion of $(2x^2 - \frac{3}{x})^{28}$. *Your answer may include unevaluated factorials, binomial coefficients, powers, products, or sums.*

Answer: $\binom{28}{15} \cdot 2^{13} \cdot (-1) \cdot 3^{15}$

Justification: $\left(2x^2 + \left(\frac{-3}{x}\right)\right)^{28} = \sum_{k=0}^{28} \binom{28}{k} (2x^2)^{28-k} \left(\frac{-3}{x}\right)^k$

$= \sum_{k=0}^{28} \binom{28}{k} 2^{28-k} \cdot (-1)^k \cdot 3^k \cdot x^{56-2k} \cdot x^{-k}$

$= \sum_{k=0}^{28} \binom{28}{k} 2^{28-k} (-1)^k \cdot 3^k \cdot \underline{x^{56-3k}}$

$56 - 3k = 11 \Rightarrow$

$3k = 45 \Rightarrow$

$k = 15$

17. Consider the following argument:

French fries are healthy, unless you put mayonnaise on them. French fries are tasty only if you put mayonnaise on them. Therefore, for french fries to be tasty it is necessary that they are unhealthy.

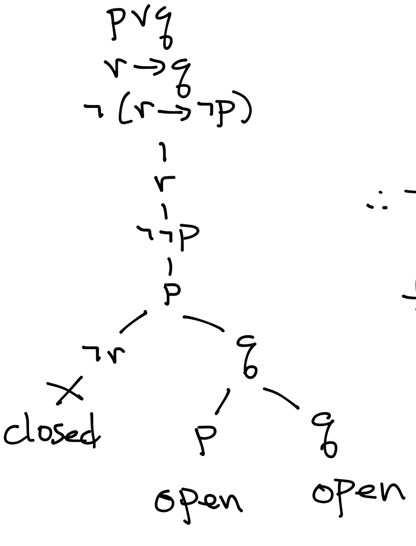
(i) Translate this argument into propositional logic. Clearly indicate the propositional variables you use.

p: fries are healthy.
q: you put mayo on fries.
r: fries are tasty.

$$\begin{array}{l} p \vee q \\ r \to q \\ \hline \therefore r \to \neg p \end{array}$$

(ii) Use a truth tree to determine the validity of the argument.

Suppose hypotheses are True & conclusion false.



∴ the valuation $v(p) = v(q) = v(r) = T$ falsifies the argument – i.e. the premisses are all <u>True</u> but the conclusion is <u>false</u>.

18. Use **Mathematical Induction** to prove that for all integers $n \geq 1$,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq \frac{n}{2} + 1.$$

$P(n)$: $1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq \frac{n}{2} + 1$

Base case $P(1)$ $1 \leq \frac{1}{2} + 1$ ✓

Ind Case Assume $P(k)$: $1 + \frac{1}{2} + \cdots + \frac{1}{k} \leq \frac{k}{2} + 1$

By IH we get $1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1} \leq \frac{k}{2} + 1 + \frac{1}{k+1}$

Now $\frac{1}{k+1} \leq \frac{1}{2}$ for $k \geq 1$, so

$$\frac{k}{2} + 1 + \frac{1}{k+1} \leq \frac{k}{2} + 1 + \frac{1}{2} = \frac{k+1}{2} + 1$$

as required. This shows $P(k+1)$.

Thus by induction $P(n)$ holds for all $n \geq 1$.

19. Define a binary relation $\mathcal{R}$ on the set $\mathbb{Z}$ as follows:

$$x\mathcal{R}y \qquad \text{if and only if} \qquad x + y = 2k \text{ for some integer } k.$$

(a) Prove that $\mathcal{R}$ is an equivalence relation.

<u>Refl</u> Take $x \in \mathbb{Z}$. Then $x + x = 2x$, so xRx.

<u>Symm</u> Suppose xRy, i.e. $x+y = 2k$ for some k.
Then $y + x = x + y = 2k$, so yRx.

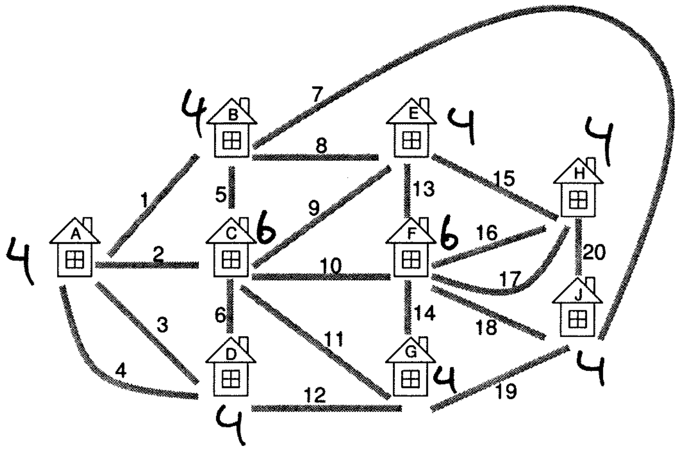
<u>Trans.</u> Suppose xRy and yRz.
Then $x + y = 2k$ for some k
$y + z = 2\ell$ for some ℓ.

Thus $x + 2y + z = 2k + 2\ell$, whence
$x + z = 2(k + \ell - y)$ so that xRz.

(b) Describe the equivalence classes of $\mathcal{R}$. How many distinct equivalence classes are there?

Any two even numbers are related.
Any two odd numbers are related.
So there are two eq. classes, n.l.
{even numbers}, {odd numbers}.

20. Consider the following village (the grey lines indicate roads between houses).



(a) Is it possible to take a walk through the village in such a way that you use every road exactly once? Cite appropriate theorems from graph theory to support your answer.

(b) Is it possible for such a walk to start and end at the same house?

(c) Suppose road number 17 is closed. Can we take a walk using all the remaining roads exactly once?

(a) + (b) : all degrees are even. Hence we can have an Euler tour, but not an Euler trail starting and ending at different houses.

(c) Now F and H have odd degree, so we can have an Euler trail starting at F and ending at H.

21. Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be arbitrary functions.

(i) Suppose that $g \circ f$ is injective. Is g then necessarily injective as well? Prove or give a counterexample.

Let $f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$ $\quad g(x) = \begin{cases} x-1 & \text{if } x > 0 \\ x & \text{if } x \leq 0 \end{cases}$.

Then $g(0) = g(1)$ so g not injective, but $g \circ f$ is the identity, so is bijective.

(ii) For a subset $A \subseteq \mathbb{Z}$, consider $f[A] = \{f(x) \mid x \in A\}$. Prove that for arbitrary $A, B \subseteq \mathbb{Z}$, we have $f[A \cup B] = f[A] \cup f[B]$.

- $f[A \cup B] \subseteq f[A] \cup f[B]$: take $y \in f[A \cup B]$.
  then $y = f(x)$, with $x \in A \cup B$.
  If $x \in A$ then $f(x) \in f[A] \subseteq f[A] \cup f[B]$
  If $x \in B$ then $f(x) \in f[B] \subseteq f[A] \cup f[B]$
  Either way, $y \in f[A] \cup f[B]$.

- $f[A] \cup f[B] \subseteq f[A \cup B]$: take $y \in f[A] \cup f[B]$.
  If $y \in f[A]$, then $y = f(x)$, $x \in A$.
  So $x \in A \cup B$, and $f(x) \in f[A \cup B]$.
  If $y \in f[B]$, then $y = f(x)$, $x \in B$.
  So $x \in A \cup B$, and $f(x) \in f[A \cup B]$.

  Either way $y = f(x) \in f[A \cup B]$.