Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Discrete Mathematics for Computing MAT1348B

Midterm Examination

2 March 2015

Prof. Philip Scott

Version B

INSTRUCTIONS:

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators are not allowed.
- The exam consists of 11 questions on 9 pages. Page 9 is for additional work. *Please do not detach it.*
- Questions 1-4 are multiple-choice. You must enter the letter corresponding to each correct answer in the table preceding Question 1. No partial marks will be given for other work.
- Questions 5-9 are short-answer. You must write your final answer in the answer box and show your work below it, justifying your answer, to receive full marks.
- Questions 10-11 are long-answer. You must clearly show all relevant steps and justify your solution to receive full marks. Clearly indicate the final answer.
- Be sure to read carefully and follow the instructions for the individual problems.
- For rough work, you may use the back pages. Do not use scrap paper of your own.
- Use proper mathematical notation and terminology.
  Note: for functions, injective = one-to-one, surjective = onto.
- If you require clarification, raise your hand.
- Good luck!

Note: there were 3 versions; they are all similar — modify this version accordingly to get the answers.

Last name: Solutions to

First name: Version B

Student number: ____________________

Signature: ____________________

Question	1 − 4	5 − 6	7 − 8	9	10	11	Total
Max	4 × 2	2 + 2	2+3	4	4	5	30
Marks							

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Questions 1–4 are multiple choice. Enter the **letter** corresponding to each correct answer in the appropriate box below.

Question	1	2	3	4
Answer	B	A	D	D

[2pts] 1. Let A and B be finite non-empty sets. Which of the following statements are **false**?

(i) If $|A| > |B|$, then no function $f : A \to B$ is surjective. False

(ii) If there exists a bijection $f : A \to B$, then $|A| = |B|$. True

(iii) If $|A| \geq |B| > 1$, then every function $f : A \to B$ is surjective. False

(iv) If $|A| > |B|$, then no function $f : A \to B$ is injective. True

(v) If there exists an injective function $f : A \to B$, then $|A| \leq |B|$. True

A. only (i) **B.** (i) and (iii) **C.** only(iii) **D.** (iii) and (iv)

E. (i) and (v) **F.** (ii) and (v) **G.** None of the previous answers is correct.

Re (i), if A has more elements than B, then map $|B|$ many elements of A to B. Then the rest ($|A|-|B|$ elements) can just map to some fixed element of B.
Re (iii): clearly false. If $|B| \geq 2$, map all of A to one of B's elements — the other is not touched. This gives a function $f : A \to B$ that's not surjective.

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2. Let P be a complex proposition, and consider a completed truth tree with P at the root. (Recall, in a completed truth tree, no further rules can be applied to any formula on the tree). Which of the following statements are **true**? [2pts]

(i) If the truth tree for P has no open paths, then P is a tautology.

(ii) If the truth tree for P has no open paths, then $\neg P$ is a tautology.

(iii) If the truth tree for P has no closed paths, then P is a tautology.

(iv) Each open path corresponds to one or more counterexamples to the statement P *is a contradiction*.

(v) The number of open paths is equal to the number of counterexamples to the statement $\neg P$ *is a tautology*.

A. (ii) and (iv) **B.** (i) and (iii) **C.** only (ii) **D.** (ii) and (v)

E. (iv) and (v) **F.** (iii) and (iv) **G.** None of the previous answers is correct.

See next page

(ii) To check if $\neg P$ is a tautology, put $\neg\neg P$ (equivalently P) at the top of the tree & draw the truth tree. If all branches are closed, then $\neg P$ is a tautology. Hence if the truth tree for $\neg P$ has no open paths, then all paths (= branches) are closed, so $\neg P$ is a tautology

(iv). An open path (branch) has the property that every formula on the path is true (by a valuation which makes all literals on the path T). In particular, the root P will be True. So this open branch gives one or more valuations v satisfying $v(P) = T$. (Sometimes an open path corresponds to more than one valuation if some atoms on the open path are missing.). In particular each open path corresponds to one or more valuations which are counterexamples to the statement that $P \equiv F$.

3. Consider the following three compound propositions:

$$P:\ (a \to b) \to c, \qquad Q:\ a \to (b \to c), \qquad \text{and} \qquad R:\ (a \wedge b) \to c$$

[2pts] Which of the propositions P, Q, and R are **equivalent**?

A. None of them. **B.** only P and Q **C.** only P and R **D.** only Q and R

E. All of them.

$$((a \wedge b) \to c) \equiv (a \to (b \to c))$$

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4. The truth table of a compound proposition p with atomic propositions A, B, and C is as follows:

Remember we mentioned many times in class and DGD that DNF's are not unique. But you could easily test other possible answers by plugging them into the truth table.

A	B	C	p	
T	T	T	T	← $A \wedge B \wedge C$
T	T	F	T	← $A \wedge B \wedge \neg C$
T	F	T	F	
T	F	F	F	
F	T	T	F	
F	T	F	T	← $\neg A \wedge B \wedge \neg C$
F	F	T	F	
F	F	F	F	

[2pts] Which of the following propositions is/are **disjunctive normal forms** of p?

(i) $(A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C)$

(ii) $(A \wedge B \wedge C) \vee (B \wedge \neg C)$

(iii) $(A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C)$

(iv) $(A \vee B \vee C) \wedge (A \vee B \vee \neg C) \wedge (\neg A \vee B \vee \neg C)$

(v) $(A \wedge B) \vee (\neg A \wedge B \wedge \neg C)$

(vi) $A \vee B \vee C$

A. (i), (iii), and (v) **B.** (iii), (iv), and (v) **C.** only (iii) **D.** (ii), (iii), and (v)

E. only (iii) and (v) **F.** None of the previous answers is correct.

ANS. (iii) is obvious. (ii) and (v) are obtained from (iii) either by examining the truth table and seeing they give the same value, alternatively, use Boolean algebra; for example for (ii):

$$\text{(iii)} = (A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \equiv (A \wedge B \wedge C) \vee [(A \vee \neg A) \wedge (B \wedge \neg C)]$$
$$\equiv (A \wedge B \wedge C) \vee [T \wedge (B \wedge \neg C)] \equiv \text{(ii)}$$

In each of the following five questions, write your final answer in the answer box.

To receive full marks, you must show your work, justifying the answer.

[2pts] 5. Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ and $B = \{0, \{0, \{1\}\}\}$. What is the **cardinality of the power set of** $A \times B$?

$|\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^{3 \times 2} = 2^6 = 64$

Here, $|A| = 3$, $|B| = 2$

Note: many students had trouble finding the cardinalities of A and B.

[2pts] 6. On the Island of Knights and Knaves, there are two types of natives, indistinguishable by sight: knights, who always tell the truth, and knaves, who always lie.

Strolling on the island, we meet two inhabitants A and B. Person B says: "A is a knight if and only if I am a knave." What is person A?

Answer: A is a Knave

B says "A Knight $\leftrightarrow$ B Knave"

Case 1: B Knight. $\therefore$ Statement is T. But "B Knave" is F in this case. Hence the LHS "A Knight" must be F (since "F $\leftrightarrow$ F" is T).

$\therefore$ A is a Knave.

Case 2: B knave. $\therefore$ statement is F. But "B Knave" is T. So in order for the statement to be F, "A knight" is F.

$\therefore$ A is a Knave.

Note: this was straightforward Knights & Knaves. There are 2 cases, and neither case is contradictory: They each give the unique answer that A is a Knave (in this version!).

[2pts] 7. Define the following atomic propositions:

N: "Parking regulations are not very strict."

P: "Parisians question parking regulations."

Y: "Yves is the mayor of Paris."

I: "Traffic in Paris improves."

Translate the following sentence into Propositional Calculus using propositional variables N, P, Y, and I:

Traffic in Paris improves and Parisians do not question parking regulations
if and only if
parking regulations being very strict is a necessary condition for Yves to be the mayor of Paris.

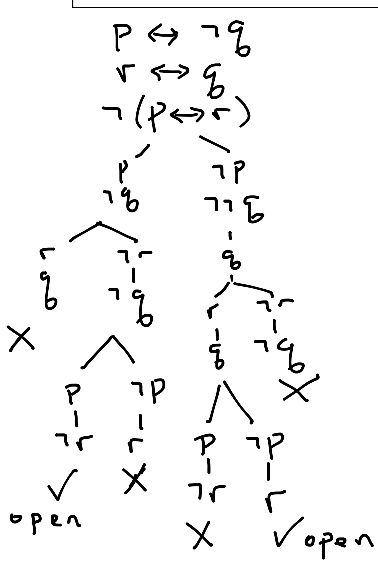
Propositional Formula: $(I \wedge \neg P) \leftrightarrow (Y \to \neg N)$

[3pts] 8. Is the following argument valid? If not, give a counterexample.

$$\begin{array}{l} p \leftrightarrow \neg q \\ r \leftrightarrow q \\ \hline \therefore p \leftrightarrow r \end{array}$$

Answer: The argument is (circle): valid invalid

Counterexample (if applicable): Must give one of two falsifying valuations



2 open paths (marked by "✓")

$V_1(p) = T$ $\quad V_2(p) = F$

$V_1(q) = V_1(r) = F$ $\quad V_2(q) = V_2(r) = T$.

Notice: for each valuation V_i $(i = 1,2)$ the premisses of the argument are True and the conclusion is false. ∴ The argument fails.

[4pts] 9. Consider the following functions:

1. $f : A \to A$, where $A = \{0, 1, 2, 3\}$, defined by $f(0) = 3,\ f(1) = 3,\ f(2) = 1,\ f(3) = 2.$
2. $g : \mathbb{Q}^2 \to \mathbb{Q}^2$ defined by $g(x, y) = (y, x - y)$.
3. $h : [0, \infty) \to \mathbb{R}$ defined by $h(x) = 3x^2 - 2$, where $[0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$.
4. $\ell : \mathbb{Z}\times\mathbb{Z}^+ \to \mathbb{R}$ defined by $\ell(x, y) = \frac{3x}{y}$, where $\mathbb{Z}^+ = \{1, 2, 3, 4, \cdots\}$ is the positive integers.

Which of these functions are injective? Which are surjective?

injective functions: g, h

surjective functions: g

1 $f(0)=f(1)$ $\therefore$ Not injective. $0 \notin \text{Range}(f)$. $\therefore$ Not surjective.

2 Injective & Surjective

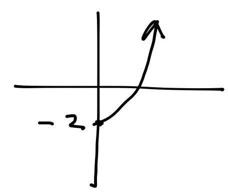
Injective : suppose $g(x,y)=g(x',y')$. $\therefore (y, x-y) = (y', x'-y')$. Hence $y=y'$.
Also $x-y = x'-y' = x'-y$. $\therefore x = x'$.

Surjective : suppose $(u,v) \in \mathbb{Q}^2$. We must find $(x,y)\in\mathbb{Q}^2$ such that $g(x,y) = (u,v)$. Setting up the equations:

$$g(x,y) = (y, x-y) = (u,v).$$

$\therefore y = u$ and $x - y = v$. $\therefore x = y+v = u+v$.
$\therefore$ let $(x,y) = (u+v, u)$. Then one easily checks:
$g(x,y) = g(u+v, u) = (u, (u+v)-u) = (u,v)$ ✓

3 . h is injective, not surjective. The graph of h is



h is increasing, hence injective (since domain $= \mathbb{R}^{\geq 0}$).
Since $h(x) \geq -2$, clearly any $y < -2$ is not in the range of h; for example, $-3 \notin \text{range}(h)$. $\therefore$ h not surjective.

4 $\frac{3\cdot 1}{1} = \frac{3 \cdot 2}{2}$ $\therefore \ell(1,1) = \ell(2,2)$. $\therefore$ Not injective.

Notice: $\text{Range}(\ell) = \mathbb{Q}$, because every rational has the form $\frac{3x}{y}$ for some $x, y \in \mathbb{Z}$, $y \neq 0$. But we know $\sqrt{2} \notin \mathbb{Q}$. $\therefore \sqrt{2} \notin \text{Range}(\ell)$.
So ℓ is neither injective nor surjective.

[4pts] 10. Let q be a rational number and r an irrational number. Using a **proof by contradiction**, show that $2q^2 + 3r$ is irrational.

Proof: By the method of contradiction.
Suppose $2q^2+3r$ is rational. Say $2q^2+3r = q' \in \mathbb{Q}$.
$\therefore$ $3r = q'-2q^2$, so $r = \frac{q'-2q^2}{3}$. But we know:
if q is rational, so is q^2 and $-2q^2$. We also know that the difference of two rationals is rational and dividing a rational by 3 still gives a rational.
Hence $r = \frac{q'-2q^2}{3}$ is rational. Contradiction.

Note: The above facts about $\mathbb{Q}$ were proved in class:

$$\frac{m}{n} + \frac{m'}{n'} = \frac{mn' + m'n}{nn'}$$

$$\frac{m}{n} \cdot \frac{m'}{n'} = \frac{mm'}{nn'}$$

A lot of students missed the general form of a proof by contradiction. It works as follows: to prove P, suppose ¬P and derive a contradiction. Hence P is true.

11. Let S and T be two sets. Consider the following two statements; each one is either true or false. If a statement is true, give a rigorous proof. If a statement is false, give a counterexample, using the universal set $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. *Fully justify your answers.*

[5pts]

(a) $\overline{\overline{S} \cap T} = \overline{S} \cup \overline{T}$

By de Morgan, $\overline{\overline{S} \cap T} = \overline{\overline{S}} \cup \overline{T} = S \cup \overline{T} \neq \overline{S} \cup \overline{T}$, in general.

Say $S = \{1,2\}$, $\overline{S} = \{3,4,5,6,7,8,9,10\}$
$T = \{1,2,3\}$ $\overline{T} = \{4,5,6,7,8,9,10\}$

$S \cup \overline{T} = \{1,2,4,5,6,7,8,9,10\}$.
$\overline{S} \cup \overline{T} = \{3,4,5,6,7,8,9,10\}$
clearly $\neq$.

(b) $S \subseteq T$ implies $S \subseteq S \cap T$.

Suppose $S \subseteq T$. Prove $S \subseteq S \cap T$.
Let $x \in S$. Then since $S \subseteq T$, $x \in T$. Hence, if $x \in S$ then $x \in S \wedge x \in T$. Hence $x \in S \cap T$. So $S \subseteq S \cap T$. ✓

Remarks: Many students wrote a correct S, T but then didn't properly calculate $S \cup \overline{T}$ and $\overline{S} \cup \overline{T}$ — too many tried to do it in their heads, instead of writing out the details.

a lot of students forgot the Definitions: $X \subseteq Y$ means $\forall x (x \in X \rightarrow x \in Y)$. $X \cap Y$ is $\{x \mid x \in X \wedge x \in Y\}$, etc.