Solution - Version 1

Université d'Ottawa • University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Discrete Mathematics for Computing - MAT 1348 B

Midterm Examination

16 February 2012
Instructor: Laura Dumitrescu

Instructions:

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators (without graphing or programming function) are allowed, but not needed.
- The exam consists of 11 questions on 10 pages. Page 10 is for additional work. Please do not detach it.
- Questions 1-6 are multiple-choice. You must enter the letter corresponding to each correct answer in the table preceding Question 1. No partial marks will be given for other work.
- Questions 7-11 are long-answer. You must clearly show all relevant steps in your solution to receive full marks. Clearly indicate the final answer.
- Be sure to read carefully and follow the instructions for the individual problems.
- For rough work, you may use the back pages. Do not use scrap paper of your own.
- Use proper mathematical notation and terminology.
- If you require clarification, raise your hand.
- Good luck!

Last name:____________________

First name:____________________

Student number:____________________

Question	1 − 6	7	8	9	10	11	Total
Max	12	3	4	5	4	5	33
Marks							

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Questions 1–6 are multiple choice. Enter the **letter** corresponding to each correct answer in the appropriate box below.

Question	1	2	3	4	5	6
Answer	B	F	E	E	A	A

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1. The truth table of a compound proposition p with atomic propositions A, B, and C is as follows:

A	B	C	p
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

Only one of the following propositions is a **disjunctive normal form** of p — which one?

A. $(A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C)$
B. $(A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge \neg C)$
C. $\neg A \wedge \neg B \wedge \neg C$
D. $(A \vee \neg B \vee C) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg A \vee \neg B \vee \neg C)$
E. $\neg A \vee \neg B \vee \neg C$
F. $(A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge \neg C)$

$(A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)$

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2. Let A and B be finite sets with $|A| = 5$ and $|B| = 2$. What is the **cardinality of the power set of** $A \times B$?

A. 4 **B.** 10 **C.** 16 **D.** 32 **E.** 512 **F.** 1024.

$|A \times B| = |A| \cdot |B| = 2 \cdot 5 = 10$

$|\mathcal{P}(A \times B)| = 2^{10} = 1024$

3. Which of the following statements are **true**?

F (i) The compound proposition $(a \to b) \to b$ is a tautology.
T (ii) If the set of premises of an argument is inconsistent, then the argument is valid.
T (iii) If X is false, Y is true, and Z is false, then $X \wedge Y \to Z$ is true.
F (iv) The compound propositions $\neg((a \to b) \to c)$ and $a \wedge b \wedge \neg c$ are logically equivalent.

A. only (iii) **B.** only (iv) **C.** only (i) **D.** (i) and (iii)

E. (ii) and (iii) **F.** only (ii)

(i)

a	b	$a \to b$	$(a \to b) \to b$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

(ii) if $\{H_1, \ldots, H_n\}$ is inconsistent, then $H_1 \wedge \ldots \wedge H_n$ is F and so $(H_1 \wedge \ldots \wedge H_n) \to C$ is T

(iii) Since $X \wedge Y$ is F, $X \wedge Y \to Z$ is T

(iv) $\neg((a \to b) \to c) \equiv \neg(\neg(a \to b) \vee c) \equiv \neg(\neg(\neg a \vee b) \vee c) \equiv (\neg a \vee b) \wedge \neg c \equiv \neg a \vee b \wedge \neg c$

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4. On the Island of Knights and Knaves you meet two inhabitants A and B. Person B says: "A is a knave only if I am a knave." Which of the following statements is **true**?

(i) A is a knight and B is a knave.
(ii) A is a knave and B is a knight.
(iii) A and B are both knaves.
(iv) A and B are both knights.
(v) B is a knight but it is impossible to determine what A is.
(vi) A is a knight but it is impossible to determine what B is.

A. (v) **B.** (iii) **C.** (i) **D.** (vi) **E.** (iv) **F.** (ii)

p: "A is a knight", q: "B is a knight"

B says: $\neg p \to \neg q$

p	q	$\neg p \to \neg q$
T	T	T ←
T	F	T
F	T	F
F	F	T

q and $\neg p \to \neg q$ need to have the same truth value

So, p is T and q is T:

A and B are both knights.

5. Let $S = \{1, \{2\}, \{1,2\}, \emptyset\}$. Which of the following statements are **true**?

(i) $\{\{1\}, \emptyset\} \subseteq S$ F since $\{1\} \notin S$ ($\{1\}$ is not an element of S)

(ii) $\{1, \{2\}\} \in S$ F since $\{1, \{2\}\} \notin S$ ($\{1, \{2\}\}$ is not an element of S)

(iii) $\{1, \{1,2\}\} \subseteq S$ T since $1 \in S$ and $\{1,2\} \in S$

(iv) $\{1,2\} \subseteq S$ F since $2 \notin S$ (2 is not an element of S)

(v) The cardinality of the power set of S is 8. F since $(|S| = 4$ and so $|\mathcal{P}(S)| = 2^4 = 16)$

(vi) $\{\emptyset\} \in S$ F since $\{\emptyset\}$ is not an element of S

A. only (iii) **B.** (i) and (iii) **C.** only (v) **D.** (ii) and (vi)

E. (iii) and (v) **F.** (iv) and (vi)

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6. Which of the following arguments (rules of inference) are **invalid**?

(i) $\begin{array}{l} a \to b \\ \neg a \\ \hline \therefore \neg b \end{array}$ (ii) $\begin{array}{l} a \to b \\ \neg b \\ \hline \therefore \neg a \end{array}$ (iii) $\begin{array}{l} a \vee b \\ \neg a \vee c \\ \hline \therefore b \vee c \end{array}$

(iv) $\begin{array}{l} a \vee b \\ \neg b \\ \hline \therefore a \end{array}$ (v) $\begin{array}{l} a \vee b \\ \neg a \vee c \\ \hline \therefore b \wedge c \end{array}$ (vi) $\begin{array}{l} a \to b \\ \neg a \to c \\ \hline \therefore \neg b \to c \end{array}$

A. (i) and (v) **B.** (ii) and (v) **C.** (iii) and (vi) **D.** (i) and (iv) **E.** only (vi)

(i) is invalid

(ii) is valid : Modus tollens

(iii) is valid : Resolution

(iv) is valid : Disjunctive syllogism

(v) is invalid

(vi) is valid: $a \to b \equiv \neg b \to \neg a$ and

$\begin{array}{l} \neg b \to \neg a \\ \neg a \to c \\ \hline \therefore \neg b \to c \end{array}$ is valid $\rightsquigarrow$ Hypothetical syllogism

3 points

7. Let A, B, and C be subsets of the universal set U. Use properties of set operations and set identities to show the following. *You need not name the identities used.*

$$A - (\overline{C} \cap B) = (A \cap C) \cup (A - B)$$

$$\begin{aligned} A - (\overline{C} \cap B) &= A \cap \overline{(\overline{C} \cap B)} && 1p \\ &= A \cap (C \cup \overline{B}) && 1p \\ &= (A \cap C) \cup (A \cap \overline{B}) && 1p \\ &= (A \cap C) \cup (A - B) \end{aligned}$$

(4 points) 8. Define the following atomic propositions:

H: "The tiger hides."
F: "The hunt is finished soon."
K: "The tiger is killed."
E: "The hunter is eaten by the tiger."
N: "The hunt is happening at night."

Translate each of the following sentences into compound logical propositions using the atomic propositions H, F, K, E, and N as defined above.

(a) For the hunt to be finished soon, it is necessary that the tiger be killed or the hunter be eaten by the tiger.

1P

$$F \longrightarrow K \vee E$$

(b) The tiger hides only if the hunt is happening at night.

1P

$$H \longrightarrow N$$

(c) For the hunt to be finished soon, it is necessary and sufficient that the hunt be happening at night and the hunter be eaten by the tiger.

1P

$$F \longleftrightarrow N \wedge E$$

(d) If the tiger hides or the hunt is not happening at night, then (the hunt is not finished soon unless the hunter is eaten by the tiger).

1P

$$H \vee \neg N \longrightarrow (\neg E \rightarrow \neg F)$$

5 points

9. Use any method you know to determine whether or not the argument below is valid. If the argument is not valid, give a counterexample.

H_1 $\quad \neg(B \wedge \neg A)$

H_2 $\quad A \leftrightarrow \neg(B \rightarrow A)$

H_3 $\quad C \leftrightarrow B$

C $\quad \therefore \neg(A \rightarrow C)$

Construct a truth tree for

$H_1 \wedge H_2 \wedge H_3 \wedge \neg C$

$\neg(B \wedge \neg A) \equiv \neg B \vee A$ ✓

$A \leftrightarrow \neg(B \to A) \equiv (A \wedge \neg(B \to A)) \vee (\neg A \wedge (B \to A))$

$\equiv (A \wedge \neg(\neg B \vee A)) \vee (\neg A \wedge (\neg B \vee A))$

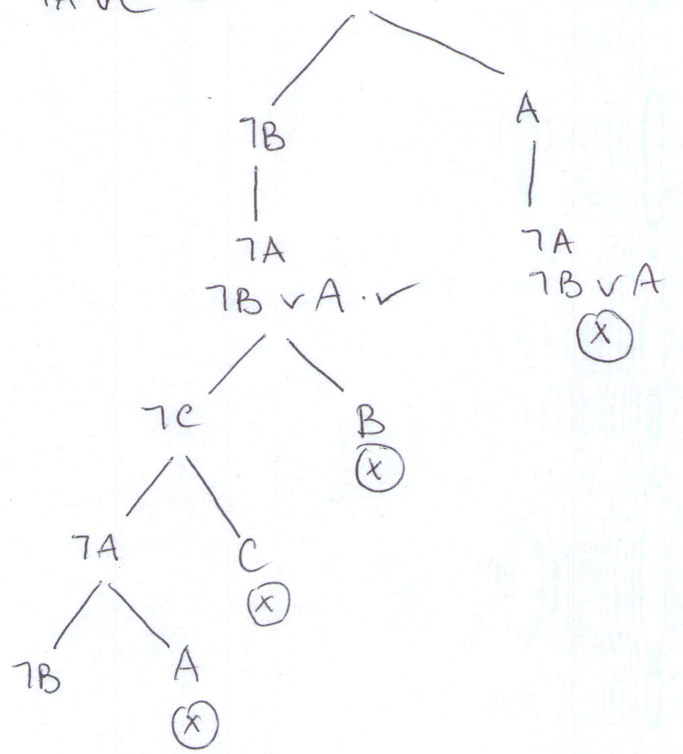
$\equiv (A \wedge B \wedge \neg A) \vee (\neg A \wedge (\neg B \vee A))$

$\equiv F \vee (\neg A \wedge (\neg B \vee A))$

$\equiv \neg A \wedge (\neg B \vee A)$ ✓

✓ $\neg C \vee B \equiv C \leftrightarrow B$

✓ $\neg A \vee C \equiv \neg\neg(A \to C)$



1p for the root of the tree

3p for the branches.

Since there exists a complete active path, the argument is not valid

Counterexample : A : F, B : F and C : F

1p

(you can verify that H_1, H_2 and H_3 are T, but C is F in this case)

(4 points) 10. Let n be an integer. Give an **indirect proof** of the following theorem.

If $n^2 + 2n - 1$ is an odd integer, then n is an even integer.

Let p: " $n^2 + 2n - 1$ is odd "

q: " n is even "

1p

We need to prove $p \to q$.

Assume $\neg q$ is true. We need to have $\neg p$ is true.

1p " n is odd "

If n is odd, there exists an integer k such that

$$n = 2k + 1$$

1p

Then,
$$\begin{aligned} n^2 + 2n - 1 &= (2k+1)^2 + 2(2k+1) - 1 \\ &= 4k^2 + 4k + 1 + 4k + 2 - 1 \\ &= 4k^2 + 8k + 2 \\ &= 2(2k^2 + 4k + 1) \end{aligned}$$

1p

If we denote $m = 2k^2 + 4k + 1$, then $n^2 + 2n - 1 = 2m$, with $m \in \mathbb{Z}$. Hence, $\neg p$ is true ($n^2 + 2n - 1$ is even).

(Please give full points even if they do not work with propositional variables, in case their mathematical proof is correct)

5 points

11. Is the function $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by

$$F(x, y) = (2y, 3x + y)$$

(a) one-to-one?

(b) onto?

(c) a bijection?

Fully justify your answer.

(a) Let $(x_1, y_1) \in \mathbb{Z} \times \mathbb{Z}$ and $(x_2, y_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $F(x_1, y_1) = F(x_2, y_2)$

1p

(This means $x_1 \in \mathbb{Z}$, $x_2 \in \mathbb{Z}$, $y_1 \in \mathbb{Z}$ and $y_2 \in \mathbb{Z}$)

So, $(2y_1, 3x_1 + y_1) = (2y_2, 3x_2 + y_2)$ and hence

$$\begin{cases} 2y_1 = 2y_2 & \Rightarrow y_1 = y_2 \\ 3x_1 + y_1 = 3x_2 + y_2 & \end{cases} \Bigg\rangle\ x_1 = x_2.$$ Therefore

1p

$(x_1, y_1) = (x_2, y_2)$,

i.e. F is one-to-one

(b) Let $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $F(x, y) = (a, b)$.

1p

It follows that $(2y, 3x + y) = (a, b)$. Solve for x and y.

$$\begin{cases} 2y = a \\ 3x + y = b \end{cases} \Rightarrow y = \frac{a}{2} \text{ and } x = \frac{b - y}{3} = \frac{b - \frac{a}{2}}{3} = \frac{b}{3} - \frac{a}{6}$$

However, there exist ~~no~~ integer values of a and b such that $(x, y) \notin \mathbb{Z} \times \mathbb{Z}$.

1p

Take $a = 1$ and $b = 1$. We have $(1, 1) \in \mathbb{Z} \times \mathbb{Z}$ but the equation $F(x, y) = (1, 1)$ does not have a solution in $\mathbb{Z} \times \mathbb{Z}$ $\left(x = \frac{1}{3} - \frac{1}{6} = \frac{1}{6} \notin \mathbb{Z} \text{ and } y = \frac{1}{2} \notin \mathbb{Z} \right)$

Therefore, F is <u>not</u> onto.

1p (c) F is <u>not</u> a bijection since F is not onto.

Additional work space. Do not detach this page.