FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION

FALL 2011 | **MATH 211 (L 04/06)** | Time: 3 hours

I.D. NUMBER	SURNAME	OTHER NAMES
	KEY	THE ANSWER

STUDENT IDENTIFICATION

Each candidate must sign the Seating List confirming presence at the examination. All candidates for final examinations are required to place their University of Calgary student I.D. cards on their desks for the duration of the examination. (Students writing mid-term tests can also be asked to provide identity proof.) Students without an I.D. card who can produce an acceptable alternative I.D., e.g., one with a printed name and photograph, are allowed to write the examination.

A student without acceptable I.D. will be required to complete an Identification Form. The form indicates that there is no guarantee that the examination paper will be graded if any discrepancies in identification are discovered after verification with the student's file. A student who refuses to produce identification or who refuses to complete and sign the Identification Form is not permitted to write the examination.

EXAMINATION RULES

1. Students late in arriving will not normally be admitted after one-half hour of the examination time has passed.
2. No candidate will be permitted to leave the examination room until one-half hour has elapsed after the opening of the examination, nor during the last 15 minutes of the examination. All candidates remaining during the last 15 minutes of the examination period must remain at their desks until their papers have been collected by an invigilator.
3. All enquiries and requests must be addressed to supervisors only.
4. Candidates are strictly cautioned against:

   (a) speaking to other candidates or communicating with them under any circumstances whatsoever;

   (b) bringing into the examination room any textbook, notebook or memoranda not authorized by the examiner;

   (c) making use of calculators and/or portable computing machines not authorized by the instructor;

   (d) leaving answer papers exposed to view;

   (e) attempting to read other students' examination papers.

   The penalty for violation of these rules is suspension or expulsion or such other penalty as may be determined.
5. Candidates are requested to write on both sides of the page, unless the examiner has asked that the left half page be reserved for rough drafts or calculations.
6. Discarded matter is to be struck out and not removed by mutilation of the examination answer book.
7. Candidates are cautioned against writing in their answer books any matter extraneous to the actual answering of the question set.
8. The candidate is to write his/her name on each answer book as directed and is to number each book.
9. A candidate must report to a supervisor before leaving the examination room.
10. Answer books must be handed to the supervisor-in-charge promptly when the signal is given. Failure to comply with this regulation will be cause for rejection of an answer paper.
11. If a student becomes ill or receives word of domestic affliction during the course of an examination, he/she should report at once to the Supervisor, hand in the unfinished paper and request that it be cancelled. Thereafter, if illness is the cause, the student must go directly to University Health Services so that any subsequent application for a deferred examination may be supported by a medical certificate. An application for Deferred Final Examinations must be submitted to the Registrar by the date specified in the University Calendar.
    Should a student write an examination, hand in the paper for marking, and later report extenuating circumstances to support a request for cancellation of the paper and for another examination, such request will be denied.
12. SMOKING DURING EXAMINATIONS IS STRICTLY PROHIBITED.

Question	Total Marks	Actual Marks
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

CALCULATORS ARE NOT ALLOWED

[10] 1. Solve the system:

$$\begin{array}{ccccccccccc} x & + & 2y & + & 2z & + & u & + & w & = & 2 \\ 2x & + & 4y & + & 4z & + & u & + & 4w & = & 7 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 1 & 2 \\ 2 & 4 & 4 & 1 & 4 & 7 \end{array}\right] \underset{R_2-2R_1}{\sim} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 & 2 & 3 \end{array}\right] \underset{-R_2}{\sim}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -2 & -3 \end{array}\right] \overset{R_1-R_2}{\sim} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -2 & -3 \end{array}\right]$$

x, u leading variables

$x = 5 - 2r - 2s - 3t$

$y = r$

$z = s$

$u = -3 + 2t$

$w = t$

$$\begin{bmatrix} x \\ y \\ z \\ u \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

[10] 2. Let $A^{-1} = GFE$ where $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$, $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.
Note that G, E and F are elementary matrices

(a) Find A.

$$A^{-1} = GFE \Rightarrow A = E^{-1}F^{-1}G^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) Find $\det A$.

$$\det A = \begin{vmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} = 1(0-(-2)) = 2$$

(c) Solve the system $AX = B$ where $B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

A square and $\det A \neq 0 \Rightarrow$ Unique solution $X = A^{-1}B$

$$X = GFE\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = GF\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = G\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1/2 \end{bmatrix}$$

(Recall $A^{-1} = GFE$ where G, F, E are elementary matrices)

[10] 3. Let A, B and C be 3×3 matrices so that $\det A = \det B \neq 0$ and $\det C = 2$.

(a) Find $\det\left(2A^{-1}C^2B^T\right)^T$.

$$\det\left[\left(2A^{-1}C^2B^T\right)^T\right] = \det\left(2A^{-1}C^2B^T\right) = 2^3\det(A^{-1})\det(C^2)\det(B^T)$$

$$= 8\frac{1}{\det(A)}(\det C)^2\det(B) = 8(\det C)^2$$

$$= 8(2^2)$$

$$= 32$$

(b) Find $\det\left(C^{-1} + adj\,C\right)$.

$$\left(\text{Note: } C^{-1} = \frac{1}{\det C}\,\text{adj}\,C = \frac{1}{2}\,\text{adj}\,C \Rightarrow \text{adj}\,C = 2C^{-1}\right)$$

$$\det(C^{-1} + \text{adj}\,C) = \det(C^{-1} + 2C^{-1}) = \det(3C^{-1}) = 3^3\det(C^{-1})$$

$$= 27\frac{1}{\det C} = \frac{27}{2}.$$

[10] 4. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix}$.

(a) Find $adj\,A$.

$$\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} -3 & -1 \\ 5 & 1 \end{vmatrix} & \begin{vmatrix} -3 & -1 \\ 5 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ -3 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -8 & 1 \\ 1 & -4 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 3 & 1 \\ -2 & -8 & -4 \\ -1 & 1 & 1 \end{bmatrix}$$

(b) Compute $A \cdot adj\,A$.

$$A \cdot \operatorname{adj} A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -2 & -8 & -4 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(c) Find $\det A$ using part (b).

$$A \cdot \operatorname{adj} A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = -2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -2I = (\det A)I$$

$$\Rightarrow \det A = -2$$

[10] 5. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Is A diagonalizable? If A is diagonalizable, find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

$$C_A(x) = \det(xI - A) = \begin{vmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{vmatrix} \overset{\substack{R_1 + xR_2 \\ R_3 - R_2}}{=} \begin{vmatrix} 0 & x^2-1 & -1-x \\ -1 & x & -1 \\ 0 & -1-x & x+1 \end{vmatrix} = \begin{vmatrix} 0 & (x+1)(x-1) & -(x+1) \\ -1 & x & -1 \\ 0 & -(x+1) & x+1 \end{vmatrix}$$

$$= -(-1)\begin{vmatrix} (x+1)(x-1) & -(x+1) \\ -(x+1) & x+1 \end{vmatrix} = (x+1)^2 \begin{vmatrix} x-1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= (x+1)^2\big((x-1)-1\big) = (x+1)^2(x-2) \Rightarrow \lambda_1 = \lambda_2 = -1, \quad \lambda_3 = 2$$

For λ_1, λ_2: $(\lambda_1 I - A)X = 0 \Rightarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow X = \begin{bmatrix} -t-s \\ s \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Take $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

For λ_3, $(\lambda_3 I - A)X = 0 \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow X_3 = \begin{bmatrix} t \\ t \\ t \end{bmatrix} \overset{t=1}{=} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

A IS DIAGONALIZABLE!

$$P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

[10] 6. Let A and B be $n \times n$ matrices. Prove the following statements:

(a) If λ is an eigenvalue of A then $\lambda^2 - 2$ is an eigenvalue of $A^2 - 2I$.

(We want to show $C_{A^2-2I}(\lambda^2-2) = 0$ given that $C_A(\lambda) = 0$)

$$C_{A^2-2I}(\lambda^2-2) = \det((\lambda^2-2)I - (A^2-2I)) = \det(\lambda^2 I - 2I - A^2 + 2I)$$
$$= \det(\lambda^2 I - A^2) = \det(\lambda I - A)\det(\lambda I + A)$$
$$= C_A(\lambda)\det(\lambda I + A) = 0 \quad \text{since } C_A(\lambda) = 0 \text{ by assumption}$$

Thus, $\lambda^2 - 2$ is an eigenvalue of $A^2 - 2I$.

(b) If $A = P^{-1}BP$ for some invertible matrix P then A and B have the same characteristic polynomial.

(We want to show $C_A(x) = C_B(x)$ given that $A = P^{-1}BP$)

$$C_A(x) = \det(xI - A) = \det(xI - P^{-1}BP) = \det(P^{-1}xIP - P^{-1}BP)$$
$$= \det(P^{-1}(xI - B)P) = \det(P^{-1})\det(xI - B)\det(P)$$
$$= \frac{1}{\det(P)} C_B(x)\det(P) = C_B(x)$$

So A and B have the same characteristic polynomial.

[10] 7. For the following, express your answers in the form $a + bi$ where a and b are real numbers.

(a) Compute $(1 - i)^{10}$.

$$|1-i| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \quad \text{so } \theta = -\pi/4.$$

Thus, $1-i = \sqrt{2}e^{-i\pi/4} \Rightarrow (1-i)^{10} = (\sqrt{2}e^{-i\pi/4})^{10} = (\sqrt{2})^{10}(e^{-i\pi/4})^{10} = 32e^{-i5\pi/2} = 32e^{-i\pi/2}$

$$= 32(\cos(-\pi/2) + i\sin(-\pi/2))$$

$$= 32(0 + i(-1))$$

$$= -32i$$

(b) Find all complex numbers z so that $z^4 = -1$.

$e^{i2k\pi} = 1$ when $k = 0,1,2,3$

$$-1 = e^{i\pi}$$

$$z^4 = -1 \Rightarrow z = (-1)^{1/4} = (e^{i\pi})^{1/4} = (e^{i\pi}e^{i2k\pi})^{1/4}, \quad k=0,1,2,3$$

$$= e^{i\frac{\pi}{4}(1+2k)}, \quad k = 0,1,2,3$$

$k=0$: $z = e^{i\frac{\pi}{4}} = \cos(\pi/4) + i\sin(\pi/4) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

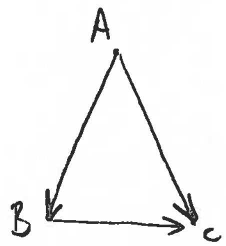
$k=1$: $z = e^{i\frac{3\pi}{4}} = \cos(3\pi/4) + i\sin(3\pi/4) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k=2$: $z = e^{i\frac{5\pi}{4}} = \cos(5\pi/4) + i\sin(5\pi/4) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$k=3$: $z = e^{i\frac{7\pi}{4}} = \cos(7\pi/4) + i\sin(7\pi/4) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

[10] 8. Consider the points $A\,(1,1,1)$, $B\,(1,2,3)$ and $C\,(2,1,3)$.

(a) Is the triangle with vertices A, B and C a right triangle? Explain.



$\vec{AB} = \vec{B} - \vec{A} = [0 \quad 1 \quad 2]^T$

$\vec{AC} = \vec{C} - \vec{A} = [1 \quad 0 \quad 2]^T$

$\vec{BC} = \vec{C} - \vec{B} = [1 \quad -1 \quad 0]^T$

$\vec{AB} \cdot \vec{AC} = 0 + 0 + 4 = 4 \neq 0$

$\vec{AB} \cdot \vec{BC} = 0 - 1 + 0 = -1 \neq 0$

$\vec{AC} \cdot \vec{BC} = 1 + 0 + 0 = 1 \neq 0$

It is <u>not</u> a right triangle

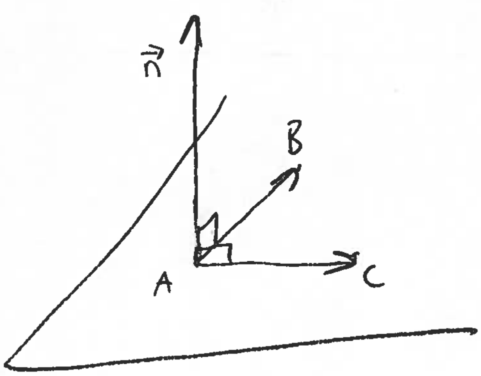
(b) Find the area of the triangle with vertices A, B and C.

$$\text{Area} = \frac{\|\vec{AB} \times \vec{AC}\|}{2}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & 0 & 1 \\ \vec{j} & 1 & 0 \\ \vec{k} & 2 & 2 \end{vmatrix} = \vec{i}(2-0) - \vec{j}(0-2) + \vec{k}(0-1) = [2 \quad 2 \quad -1]^T$$

$$\frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\sqrt{2^2 + 2^2 + (-1)^2} = \frac{1}{2}\sqrt{9} = \frac{3}{2}$$

(c) Find an equation of the plane containing the points A, B and C.



$\vec{n} = \vec{AB} \times \vec{AC} = [2 \quad 2 \quad -1]^T$

$\vec{n} \cdot [x \quad y \quad z]^T = \vec{n} \cdot \vec{A}$

$\Rightarrow 2x + 2y - z = 2(1) + 2(1) - 1$

$= 3$

EQUATION OF PLANE: $2x + 2y - z = 3$

[10] 9. Let P_1 be the plane with equation $x + y + z = 2$ and P_2 be the plane with equation $2x - y + z = 3$. Consider the point $A\,(3, 7, -2)$.

(a) Find an equation of the line that passes through the point A and is parallel to both of the planes P_1 and P_2.

$\vec{n}_1 = [1 \quad 1 \quad 1]^T$ (normal for P_1)

$\vec{n}_2 = [2 \quad -1 \quad 1]^T$ (normal for P_2)

$$\vec{d} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & 1 & 2 \\ \vec{j} & 1 & -1 \\ \vec{k} & 1 & 1 \end{vmatrix}$$

$$= \vec{i}(2) - \vec{j}(-1) + \vec{k}(-3)$$

$$= [2 \quad 1 \quad -3]^T$$

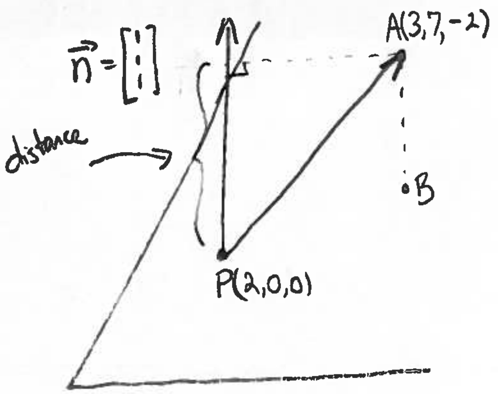
EQUATION OF LINE:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{A} + t\vec{d}$$

$$= \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

(b) Find the shortest distance between the point A and the plane P_1 and find the coordinates of the point B on P_1 that is closest to A.

Let $P = P(2,0,0)$. Then P is on the plane P_1.



$$\vec{PA} = \vec{A} - \vec{P} = \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix}$$

$$\vec{u}_1 = \text{proj}_{\vec{n}} \vec{PA} = \frac{\vec{PA} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{1+7-2}{1+1+1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{distance} = \|\vec{u}_1\| = 2\sqrt{1^2+1^2+1^2} = 2\sqrt{3}.$$

$$\vec{B} = \vec{A} - \vec{u}_1 = \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$$

$$\Rightarrow B = B(1, 5, -4)$$

[10] 10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$ and $T\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$.

(a) Find the matrix of T; that is, find a matrix A so that $T\vec{v} = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$.

$$E_1 = \begin{bmatrix}1\\0\end{bmatrix} = 2\begin{bmatrix}2\\-1\end{bmatrix} + 1\begin{bmatrix}-3\\2\end{bmatrix} \Rightarrow T(E_1) = T\left(2\begin{bmatrix}2\\-1\end{bmatrix} + 1\begin{bmatrix}-3\\2\end{bmatrix}\right) = 2T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) + 1T\left(\begin{bmatrix}-3\\2\end{bmatrix}\right)$$

$$= 2\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$$

$$E_2 = \begin{bmatrix}0\\1\end{bmatrix} = 2\begin{bmatrix}-3\\2\end{bmatrix} + 3\begin{bmatrix}2\\-1\end{bmatrix} \Rightarrow T(E_2) = T\left(2\begin{bmatrix}-3\\2\end{bmatrix} + 3\begin{bmatrix}2\\-1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}-3\\2\end{bmatrix}\right) + 3T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right)$$

$$= 2\begin{bmatrix}3\\2\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}9\\10\end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix}T(E_1) & T(E_2)\end{bmatrix} = \begin{bmatrix}5 & 9\\6 & 10\end{bmatrix}$$

(b) Is T invertible? If T is invertible, find the matrix of T^{-1}.

T is invertible if and only if A is invertible, and if so, A^{-1} is the matrix of T^{-1}.

$\det A = 50 - 54 = -4$, so $A^{-1} = \frac{1}{-4}\begin{bmatrix}10 & -9\\-6 & 5\end{bmatrix}$ is the matrix of T^{-1}.

(c) Is there a vector $\vec{a} \in \mathbb{R}^2$ so that $T\vec{a} = \begin{bmatrix}3\\2\end{bmatrix}$? If so, find $\vec{a}$.

$$T\vec{a} = \begin{bmatrix}3\\2\end{bmatrix} \Rightarrow A\vec{a} = \begin{bmatrix}3\\2\end{bmatrix} \Rightarrow \vec{a} = A^{-1}\begin{bmatrix}3\\2\end{bmatrix} = \frac{-1}{4}\begin{bmatrix}10 & -9\\-6 & 5\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}-3\\2\end{bmatrix}$$

so $\vec{a} = \begin{bmatrix}-3\\2\end{bmatrix}$

OR, you could notice that $T\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$, which shows that $\vec{a} = \begin{bmatrix}-3\\2\end{bmatrix}$.

End of Examination