UNIVERSITY OF CALGARY

SCHULICH SCHOOL OF ENGINEERING

FINAL EXAMINATION

ENGG 407 – WINTER 2007

Duration of Exam: 3 hrs

The examination is closed book and closed notes.

Scientific calculators without formula storage and/ or graphical capabilities may be used. However, communication between devices is prohibited.

Answer all five questions and show all your work in answer booklets.

Problem 1 (20 marks)

a) Use a transformation to linearize the following model

$$y = \alpha x e^{\beta x}$$

b) Estimate the values of α and β based upon regression for the following data.

x	*y*
0.1	0.75
0.2	1.25
0.4	1.45
0.6	1.25
0.9	0.85

Problem 2 (20 marks)

Given the data

x	f(x)
1.0	2.5
2.5	8.7
3.5	25.5
5.5	101.4
7.5	285.1
8.5	342.9

Use a suitable THIRD ORDER Lagrange polynomial to calculate f(4.5). Justify your choice.

Problem 3 (20 marks)

Use $O(h^6)$ Romberg integration to evaluate the following integral:

$$\int_0^{0.5} \left(10e^{-x} \sin 2\pi x\right)^2 dx$$

Estimate the approximation error ε_a for the final result.

Problem 4 **(20 marks)**

Given the first-order ODE

$$\frac{dy}{dt} = -1000\ \left[y - \left(t + 2\right)\right] + 1 \qquad y(0) = 1$$

a) Estimate maximum step size that will avoid oscillations in the numerical solution when using explicit Euler method.
b) Use the implicit Euler method to obtain a solution from $x = 0.$ to 0.1 with step size of 0.05.

Problem 5 **(20 marks)**

Solve the following boundary value problem using the Finite Difference Method (Equilibrium Method).

$$x^2 T'' + xT' = T - T_a \qquad T(1) = 0 \ \text{ and } \ T(2.0) = 2$$

Assume $\Delta x = 0.25$ and $T_a = 1$.

Write your final solution in a matrix form. There is no need to solve the matrix.

ENGG407 - Winter 2007

FORMULA SHEET FOR FINAL EXAMINATION

Taylor Series about x_i:

$$f(x) = f(x_i) + f'(x_i)(x - x_i) + \frac{f''(x_i)}{2!}(x - x_i)^2 + \frac{f^{(3)}(x_i)}{3!}(x - x_i)^3 + \ldots + \frac{f^{(n)}(x_i)}{n!}(x - x_i)^n + R_n$$

Centered approximation of second derivative:

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$$

Linear regression:

$$s_t = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

$$\begin{cases} a_1 = \dfrac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2} \\ a_0 = \bar{y} - a_1\bar{x} \end{cases}$$

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

$$s_{y/x} = \sqrt{\frac{S_r}{n-2}}$$

Lagrange Interpolating Polynomial:

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Trapezoidal rule (Multiple applications):

$$I = (b - a)\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}$$

Romberg Integration:

$$I_{j,k} = \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1} \qquad \varepsilon_a = \left| \frac{I_{j,k} - I_{j+1,k-1}}{I_{j,k}} \right| \times 100 \quad \%$$

Richardson Extrapolation:

$$D \cong \frac{4}{3} D(h_2) - \frac{1}{3} D(h_1)$$

Implicit Euler:

$$y_{i+1} = y_i + f(x_{i+1}, y_{i+1}) \times h$$

Runge Kutta Fourth Order:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_2 h) \qquad k_4 = f(x_i + h, y_i + k_3 h)$$

**************************** A. S. Z. and L. B.

Problem 1 **(25 marks)**

a) Use a transformation to linearize the following model

$$y = \alpha\, x\, e^{\beta x}$$

b) Estimate the values of α and β based upon regression for the following data.

x	*y*
0.1	0.75
0.2	1.25
0.4	1.45
0.6	1.25
0.9	0.85

Solution of Problem 1:

The function can be linearized by dividing it by *x* and taking the natural logarithm to yield

$$\ln(y/x) = \ln\alpha + \beta\ x$$

Therefore, if the model holds, a plot of ln(*y*/*x*) versus *x* should yield a straight line with an intercept of lnα and a slope of β.

xi	xi*xi	yi	Yi=Ln(yi/xi)	xi*Yi
0.1	0.01	0.75	2.01490	0.20149
0.2	0.04	1.25	1.83258	0.366516
0.4	0.16	1.45	1.28785	0.515142
0.6	0.36	1.25	0.73397	0.440382
0.9	0.81	0.85	-0.05716	-0.05144
2.2	1.38	5.55	5.81215	1.472087

$n = 5$ $\sum x_i = 2.2$ $\sum x_i^2 = 1.38$ $\sum Y_i = 5.81215$

$\sum x_i Y_i = 1.472087$

$$\bar{x} = \frac{2.2}{5} = 0.44 \qquad \bar{Y} = \frac{5.81215}{5} = 1.16243$$

$$\begin{cases} a_1 = \dfrac{n\sum x_i Y_i - \sum x_i \sum Y_i}{n\sum x_i^2 - (\sum x_i)^2} = \dfrac{5\times 1.472087 - 2.2\times 5.81215}{5\times 2.2 - 2.2^2} = -0.88089 \\ a_0 = \bar{Y} - a_1\bar{x} = 1.16243 - (-0.88089)(0.44) = 1.55002 \end{cases}$$

Therefore, $\beta = -0.88089$ and $\alpha = e^{1.55002} = 4.71156$, and the fit is

$$y = 4.71156 x e^{-0.88089x}$$

Problem 2 (25 marks)

Given the data

x	f(x)
1.0	2.5
2.5	8.7
3.5	25.5
5.5	101.4
7.5	285.1
8.5	342.9

Calculate f(4.5) using a third-order Lagrange polynomial.

Solution of Problem 2:

x	f(x)
1.0	2.5
2.5	8.7
3.5	25.5
5.5	101.4
7.5	285.1
8.5	342.9

Third order Lagrange:

$x_0 = 2.5 \quad f(x_0) = 8.7$

$x_1 = 3.5 \quad f(x_1) = 25.5$

$x_2 = 5.5 \quad f(x_2) = 101.4$

$x_3 = 7.5 \quad f(x_3) = 285.1$

$$f_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$f_3(4.5) = \frac{(4.5-2.5)(4.5-3.5)(4.5-5.5)}{(2.5-3.5)(2.5-5.5)(2.5-7.5)} 8.7 + \frac{(4.5-2.5)(4.5-5.5)(4.5-7.5)}{(3.5-2.5)(3.5-5.5)(3.5-7.5)} 25.5 + \frac{(4.5-2.5)(4.5-3.5)(4.5-7.5)}{(5.5-2.5)(5.5-3.5)(5.5-7.5)} 101.4 + \frac{(4.5-2.5)(4.5-3.5)(4.5-5.5)}{(7.5-2.5)(7.5-3.5)(7.5-5.5)} 285.1 = 53.60$$

Problem 3 (25 marks)

Use $O(h^6)$ Romberg integration to evaluate the following integral:

$$\int_0^{0.5} \left(10e^{-x}\sin(2\pi x)\right)^2 dx$$

Estimate the approximation error ε_a for the final result.

Solution of Problem 3:

n	h	K=1 $O(h^2)$ Trapezoidal	K=2 $O(h^4)$ Romberg	K=3 $O(h^6)$ Romberg
1	0.5	0.0	20.21769	
2	0.25	15.16327	15.48082	15.16503
4	0.125	15.40143		

$$\varepsilon_a = \left|\frac{15.16503 - 15.48082}{15.16503}\right| * 100\% = 2.08236\%$$

Problem 4 (20 marks)

Given the first-order ODE

$$\frac{dy}{dt} = -1000\left[y - (t+2)\right] + 1 \qquad y(0) = 1$$

c) Estimate maximum step size that will avoid oscillations in the numerical solution when using explicit Euler method.

d) Use the implicit Euler method to obtain a solution from $x = 0.$ to 0.1 with step size of 0.05.

Solution of Problem 4:

a) h<=2/1000=0.002

b)

$$y_{i+1} = y_i + \left(-1000\left[y_{i+1} - \left(t_{i+1} + 2\right)\right] + 1\right)\Delta t$$

Rearrange (Solve for y_{i+1}) :

$$y_{i+1} = \frac{1}{1 + 1000\ \Delta t}\left(y_i + 1000\ \left(t_{i+1} + 2\right)\Delta t + \Delta t\right)$$

with step size of 0.05 :

t_i	y_i
0.00	1.000000
0.05	2.030392
0.10	2.099616

Problem 5 **(20 marks)**

Solve the following boundary value problem using the Finite Difference Method (Equilibrium Method).

$$x^2 T'' + xT' = T - T_a \qquad T(1) = 0 \quad \text{and} \quad T(2.0) = 2$$

Assume $\Delta x = 0.25$ and $T_a = 1$.
Write your final solution in a matrix form. There is no need to solve the matrix.

Solution of Problem 5:

Note: the exact solution is

$$T(x) = x - \frac{2}{x} + 1 \quad (2)$$

Finite difference scheme: write

$$T'' = \frac{T_{i+1} + T_{i-1} - 2T_i}{h^2}; \qquad T' = \frac{T_{i+1} - T_{i-1}}{2h} \quad (3)$$

Then, also using $x_i = 1 + ih$,

$$(1 + ih)^2 \frac{T_{i+1} + T_{i-1} - 2T_i}{h^2} + (1 + ih)\frac{T_{i+1} - T_{i-1}}{2h} - T_i = -1 \quad (4)$$

Rearranging,

$$[2(1+ih)^2 - (1+ih)h]T_{i-1} - [4(1+ih)^2 + 2h^2]T_i + [2(1+ih)^2 + (1+ih)h]T_{i+1} = -2h^2 \quad (5)$$

Also remember for 4 segments, $h = 1/4$. First equation ($i = 1$): $T_0 = 0$ so

$$-[4(1+h)^2 + 2h^2]T_1 + [2(1+h)^2 + (1+h)h]T_2 = -2h^2 \quad (6)$$

Last equation ($i = 3$): $T_4 = 2$ so

$$[2(1+3h)^2 - (1+3h)h]T_2 - [4(1+3h)^2 + 2h^2]T_3 = -[2(1+3h)^2 + (1+3h)h] \times 2 - 2h^2 \quad (7)$$

As to the second equation ($i = 2$),

$$[2(1+2h)^2 - (1+2h)h]T_1 - [4(1+2h)^2 + 2h^2]T_2 + [2(1+2h)^2 + (1+2h)h]T_3 = -2h^2 \quad (8)$$

Replacing h and multiplying by 16,

$$-[4(4+1)^2 + 2]T_1 + [2(4+1)^2 + (4+1)]T_2 = -2 \quad (9)$$

$$[2(4+2)^2 - (4+2)]T_1 - [4(4+2)^2 + 2]T_2 + [2(4+2)^2 + (4+2)]T_3 = -2 \quad (10)$$

$$[2(4+3)^2 - (4+3)]T_2 - [4(4+3)^2 + 2]T_3 = -[2(4+3)^2 + (4+3)] \times 2 - 2 \quad (11)$$

So we need to solve

$$\begin{vmatrix} -102 & 55 & 0 \\ 66 & -146 & 78 \\ 0 & 91 & -198 \end{vmatrix} \begin{vmatrix} T_1 \\ T_2 \\ T_3 \end{vmatrix} = \begin{vmatrix} -2 \\ -2 \\ -212 \end{vmatrix} \quad (12)$$

The solution of which is $T_1 = 0.647$, $T_2 = 1.164$ and $T_3 = 1.606$. This compares with the exact solution: $T_1 = 0.650$, $T_2 = 7/6 = 1.167$, $T_3 = 45/28 = 1.607$.