8·1 Systems of linear equations

A system of equations is a collection of two or more equations, each containing one or more variables. If the collection of equations contain linear equations, the collection is called systems of linear equations.

Example :

(i) $\begin{cases} x+y=10 \\ 4x-5y=2 \end{cases}$ (ii) $\begin{cases} x+y+z=2 \\ 2x+2y=5 \\ x-y-z=10 \end{cases}$

1. Solve systems of equations by substitution

Problems

18. Solve the system of equations. If the system has no solution, say that is inconsistent.

We will use the method of substitution in this problem. In general, one can use any method.

$$\begin{cases} x+2y=-7 & \text{----- (i)} \\ x+y=-3 & \text{----- (ii)} \end{cases}$$

We have to find out the values of x and y. let us first consider the equation number (i)

$$\left.\begin{aligned} x+2y &= -7 \\ \Rightarrow 2y &= -7-x \\ \Rightarrow y &= \frac{-(7+x)}{2} \end{aligned}\right\} \text{ solve equation (i) for } y \text{ in terms of } x.$$

Now, we substitute the value of y from equation (i) to the equation (ii) as follows

$$x+y=-3$$
$$\Rightarrow x+\frac{-(7+x)}{2}=-3$$
$$\Rightarrow x-\frac{7+x}{2}=-3$$
$$\Rightarrow \frac{2x-7-x}{2}=-3$$
$$\Rightarrow 2x-7-x=-6$$
$$\Rightarrow x-7=-6$$
$$\Rightarrow x=1$$

Once, the value of x is known, one can substitute the value of x to any of the equation (i) or (ii) to obtain the value of y. Let us take the equation (ii) to obtain the value of y.

$$x+y=-3$$
$$\Rightarrow +1+y=-3$$
$$\Rightarrow y=-4$$

$\therefore$ The solutions of the system of equations (i) and (ii) are

$$\begin{cases} x=1 \\ y=-4 \end{cases}$$

One should check the solutions to confirm their findings. Substitute the values of x and y to both of the equations (i) and (ii), and, see whether they satisfy the right hand sides. For example, let us consider equation (i)

$$x+2y$$
$$\cancel{=13+2(-16)}$$
$$\cancel{=13-32}$$
$$=1+2(-4)$$
$$=1-8$$
$$=-7 \quad \text{(verified)}$$

equation (ii)

$$x+y=1-4=-3 \quad \text{(verified)}$$

So, one now believes that the solutions are correct.

2. Solve systems of equations by elimination

28. Solve the system of equations. If the system has no solution, say that it is inconsistent.

$$\begin{cases} 3x + 3y = -1 & \text{---(i)} \\ 4x + y = \frac{8}{3} & \text{---(ii)} \end{cases}$$

Here, we use the method of elimination. That means, we have to first eliminate one of the variables and then solve the other variable.

$$3x + 3y = -1 \quad \text{--(i)} \times 1$$
$$4x + y = \frac{8}{3} \quad \text{--(ii)} \times 3$$

let us, first, multiply both sides of equation (i) by 1, and, equation (ii) by 3. The results are shown below -

$$\begin{array}{rcrcl} 3x & + & \cancel{3y} & = & -1 \quad \text{--(iii)} \\ 12x & + & \cancel{3y} & = & 8 \quad \text{---(iv)} \\ (-) & & (-) & & (-) \\ \hline -9x & & & = & -9 \end{array}$$

$$\Rightarrow x = 1$$

Now, you substitute the value of x to any of the equations (i), (ii), (iii) or (iv) to obtain the value of y. let us consider equation (i) —

$$3x + 3y = -1$$
$$\Rightarrow 3 \cdot 1 + 3y = -1$$
$$\Rightarrow 3y = -4$$
$$\Rightarrow y = -\frac{4}{3}$$

$\therefore$ Solutions are

$$\begin{cases} x = 1 \\ y = \frac{-4}{3} \end{cases}$$

Of course, one should check the results as following the method, shown in the last problem.

System of dependent equations

30. Solve the system of equations. If the system has no solution, say that is inconsistent.

$$\begin{cases} 3x - y = 7 & \cdots (i) \\ 9x - 3y = 21 & \cdots (ii) \end{cases}$$

let us, first, use the method of elimination. Therefor, we multiply both sides of equation (i) by 3 and ~~both~~ equation (ii) by 1.

$$9x - 3y = 21 \quad \cdots (iii)$$
$$9x - 3y = 21 \quad \cdots (iv)$$

One should notice that equations (iii) and (iv) are actually equal to each other. That means, we have only one equation to solve in the system of equations. We follow the usual procedure to solve any of the equations of (iii) or (iv).

$$9x - 3y = 21$$
$$\Rightarrow -3y = 21 - 9x$$
$$\Rightarrow y = \frac{9x - 21}{3} = 3x - 7 \quad \cdots (v)$$

Therefor, one has infinite number of solutions for x ~~any~~ and y. If you choose the value of x to any real number, there is a corresponding value of y. The system of values are, then, one of the solutions. let us make a table to show few of the possible solutions -

x	0	1	2	-1	-2
y	-7	-4	-1	-10	-13

Inconsistent system of linear equations

26. Solve the system of equations. If the system has no solution, say that is inconsistent.

$$\begin{cases} x - y = 5 & \text{---(i)} \\ -3x + 3y = 2 & \text{---(ii)} \end{cases}$$

Let us use the method of substitution to solve the above equations (i) and (ii). First, we solve y in terms of x from equation (ii)

$$-3x + 3y = 2$$

$$\Rightarrow \quad 3y = 2 + 3x$$

$$\Rightarrow \quad y = \frac{2+3x}{3}$$

Now, substitute the value of y in equation (i)

$$x - y = 5$$

$$\Rightarrow \quad x - \frac{2+3x}{3} = 5$$

$$\Rightarrow \quad \frac{3x - 2 - 3x}{3} = 5$$

$$\Rightarrow \quad \cancel{3}x - 2 - \cancel{3}x = 15$$

$$\Rightarrow \quad 2 = 15 \quad \text{---(iii)}$$

The statement (iii) is false. Therefore the system is inconsistent and, therefore, there is no solution for the system of equations.

Solve system of three equations containing three variables

42. Solve the system of equations. If the system has no solution, say that it is inconsistent.

$$2x + y = -4 \quad \cdots (i)$$
$$-2y + 4z = 0 \quad \cdots (ii)$$
$$3x - 2z = -11 \quad \cdots (iii)$$

Use the method of elimination. First, we take equation (i) and (ii)

$$2x + y = -4 \quad \cdots (i) \times 2$$
$$-2y + 4z = 0 \quad \cdots (ii) \times 1$$

Then add them together –

$$4x + \cancel{2y} = -8$$
$$-\cancel{2y} + 4z = 0$$

$$4x + 4z = -8$$
$$\Rightarrow x + z = -2 \quad \cdots (iv)$$

Now, we perform the same work for equations (iii) and (iv)

$$3x - 2z = -11 \quad \cdots (iii) \times 1$$
$$x + z = -2 \quad \cdots (iv) \times 2$$

~~Substitute~~ Add the latter equation ~~from~~ with the former one –

$$3x - \cancel{2z} = -11$$
$$2x + \cancel{2z} = -4$$

$$5x = -15$$
$$\therefore x = -3$$

Substitute the value of x in any of the equations of (iii) and (iv) to obtain the value of z. Let us take equation (iv).

$$x + z = -2$$

$$\Rightarrow -3 + z = -2$$

$$\Rightarrow z = 1.$$

Now, substitute the value of x in equation (i) to obtain the value of y -

$$2x + y = -4$$

$$\Rightarrow 2(-3) + y = -4$$

$$\Rightarrow y = -4 + 6$$

$$= 2$$

$\therefore$ The solution set is $\begin{cases} x = -3 \\ y = 2 \\ z = 1 \end{cases}$.

Identify the ~~inconsistent~~ dependent system of linear equations containing three variables.

48. $\begin{cases} 2x - 3y - z = 0 & \text{--- (i)} \\ 3x + 2y + 2z = 2 & \text{--- (ii)} \\ x + 5y + 3z = 2 & \text{--- (iii)} \end{cases}$

Use method of elimination for equations (i) and (ii)

$$2x - 3y - z = 0 \quad \text{--- (i)} \times 2$$

$$3x + 2y + 2z = 2 \quad \text{--- (ii)} \times 3$$

$$4x - \cancel{6y} - 2z = 0$$
$$9x + \cancel{6y} + 6z = 6$$
$$13x + 4z = 6 \quad \cdots (iv)$$

Now use the same procedure for equations (ii) and (iii)

$$3x + 2y + 2z = 2 \cdots (ii) \times 5$$
$$x + 5y + 3z = 2 \cdots (iii) \times (-2)$$

$$15x + \cancel{10y} + 10z = 10$$
$$-2x - \cancel{10y} - 6z = -4$$
$$13x + 4z = 6 \quad \cdots (v)$$

You see that equation (iv) and (v) are actually same. So the equations are dependent. Effectively we have only two equations as follows.

$$13x + 4z = 6 \quad \cdots (vi)$$
$$2x - 3y - z = 0 \quad \cdots (vii)$$

(One can choose any one of the three equations (i), (ii) and (iii))

Now solve z from eqn. (vi)

$$13x + 4z = 6$$
$$\Rightarrow 4z = 6 - 13x$$
$$\Rightarrow z = \frac{6-13x}{4} \quad \cdots (viii)$$

Substitute (VIII) in (VII)

$$2x - 3y - \frac{6-13x}{4} = 0$$

$$\Rightarrow \frac{8x - 12y - 6 + 13x}{4} = 0$$

$$\Rightarrow 21x - 12y - 6 = 0$$

$$\Rightarrow 21x - 12y = 6$$

$$\Rightarrow y = \frac{21x-6}{12} = \frac{7x-2}{4} \quad \cdots (ix)$$

$\therefore$ The solutions are -

$$\begin{cases} y = \dfrac{7x-2}{4} \\ z = \dfrac{6-13x}{4} \end{cases}$$

Certainly, one has infinite many solutions.

~~Identify inconsistent systems of linear equations containing three variables~~

47. $$\begin{cases} x - y - z = 1 & \text{--- (1)} \\ -x + 2y - 3z = -4 & \text{--- (2)} \\ 3x - 2y - 7z = 0 & \text{--- (3)} \end{cases}$$

Use the method of elimination for (1) and (2)

$$\begin{array}{l} 2x - \cancel{2y} - 2z = 2 \Rightarrow \text{obtained by } 2\times(1) + 1\times(2) \\ -x + \cancel{2y} - 3z = -4 \\ \hline x - 5z = -2 \quad \text{--- (4)} \end{array}$$

Using the method of elimination for (2) and (3)

$$\begin{array}{l} -x + \cancel{2y} - 3z = -4 \\ 3x - \cancel{2y} - 7z = 0 \\ \hline 2x - 10z = -4 \quad \text{--- (5)} \end{array}$$

Now using the method of elimination in (4) and (5)

$$\begin{array}{rcl} \cancel{2x} - \cancel{10z} & = & -\cancel{4} \\ \cancel{2x} - \cancel{10z} & = & -\cancel{4} \\ - \quad + & & + \\ \hline 0 & = & 0 \end{array}$$

$\therefore$ (4) and (5) are same.

$\therefore$ Effective equations are –

$$x - 5z = -2 \quad \text{—— (6)}$$

$$x - y - z = 1 \quad \text{—— (7)}$$

Solve (6) $\Rightarrow$

$$x - 5z = -2$$

$$\Rightarrow z = \frac{2+x}{5} \quad \text{—— (8)}$$

Solve (7) after substituting z from (8)

$$x - y - \frac{2+x}{5} = 1$$

$$\Rightarrow \frac{5x - 5y - 2 - x}{5} = 1$$

$$\Rightarrow 5x - 5y - 2 - x = 5$$

$$\Rightarrow 4x - 5y = 7$$

$$\therefore y = \frac{4x-7}{5} \quad \text{—— (9)}$$

$\therefore$ The solutions are –

$$\begin{cases} y = \dfrac{4x-7}{5} \\ z = \dfrac{2+x}{5} \end{cases}$$

4A. Identify inconsistent systems of equations containing three variables

45. $$\begin{cases} x - y - z = 1 & \text{— (1)} \\ 2x + 3y + z = 2 & \text{— (2)} \\ 3x + 2y = 0 & \text{— (3)} \end{cases}$$

Use method of elimination in (1) and (2)

$$\begin{array}{l} \cancel{2x} - 2y - 2z = 2 \\ \cancel{2x} + 3y + z = 2 \\ (-) \quad (-) \quad (-) \quad (-) \\ \hline -5y - 3z = 0 \end{array}$$

$\Rightarrow 5y + 3z = 0$ — (4)

Use method of elimination in (2) and (3)

$$\begin{array}{l} \cancel{6x} + 9y + 3z = 6 \\ \cancel{6x} + 4y \quad = 0 \\ (-) \quad (-) \\ \hline 5y + 3z = 6. \end{array}$$

— (5)

Now using method of elimination in (4) and (5)

$$\begin{array}{l} \cancel{5y} + \cancel{3z} = 0 \\ \cancel{5y} + \cancel{3z} = 6 \\ (-) \quad (-) \quad (-) \\ \hline 0 = -6 \end{array}$$

$\Leftarrow$ The statement is false.

The system of equations (i), (2) and (3) are therefore inconsistent.

8.6 Systems of non-linear equations

Solving system of nonlinear equations using substitution

7. Graph each equation of the system. Then solve the system to find the points of intersection.

$$\begin{cases} y = \sqrt{36-x^2} & \text{---- (i)} \\ y = 8-x & \text{---- (ii)} \end{cases}$$

let us first analyse the equation (i)

$$y = \sqrt{36-x^2}$$

$$\Rightarrow y^2 = 36-x^2 \quad \text{(Squaring both sides)}$$

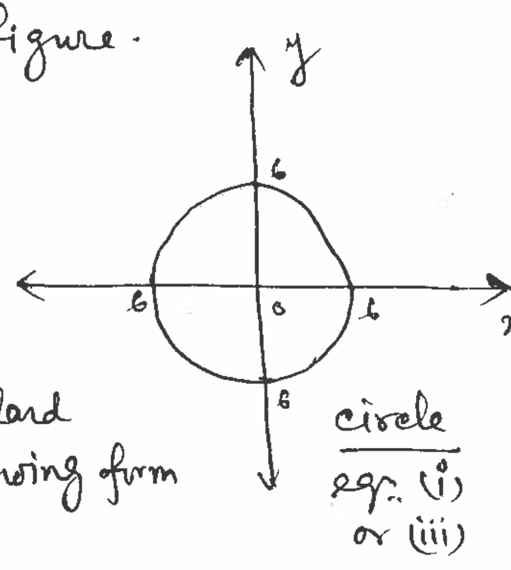
$$\Rightarrow x^2+y^2 = 36 \quad \text{---- (iii)}$$

Remember, the equation of a circle is written in the following form $(x-h)^2+(y-k)^2=a^2$, where (h,k) is the center and a is the radius of the circle.

Comparing the equation of the circle with equation (iii), one obtains

$$(h,k) = (0,0) \quad \text{and} \quad a=6.$$

Therefore, the equation (i) or (iii) is a circle of radius 6, whose center is at $(0,0)$, as shown in the figure.



circle eq. (i) or (iii)

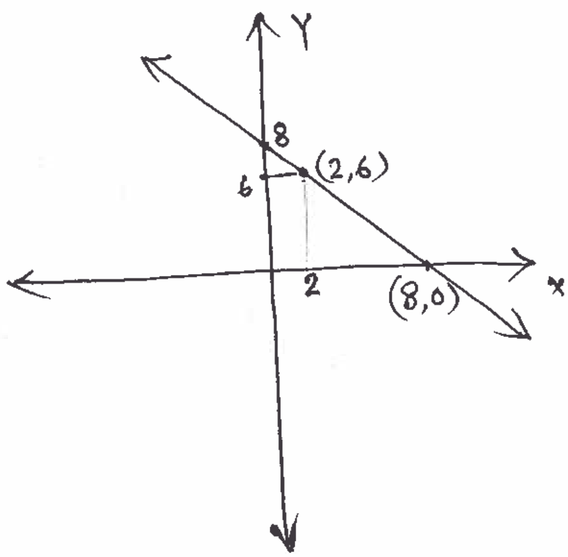
let us now look at the equation (ii).
We can write the equation in the following form

$$y = (-1)x + 8 \quad \text{--- (iv)}$$

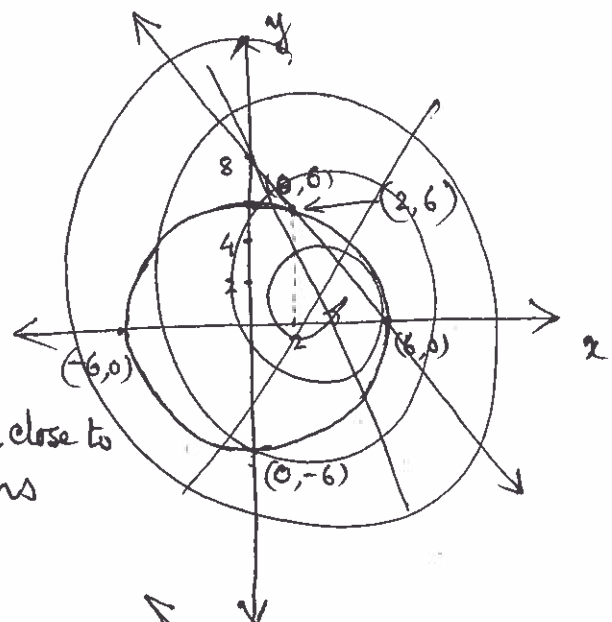
Now compare equation (iv) with the standard form of the equation of a line in the following form

$y = mx + c$, where m is the slope and c is the intercept of the line with y-axis.

Therefore, equation (ii) or (iv) is a straight line of slope (-1) and having an intercept $= 8$ in the y-axis. The graph is shown below.

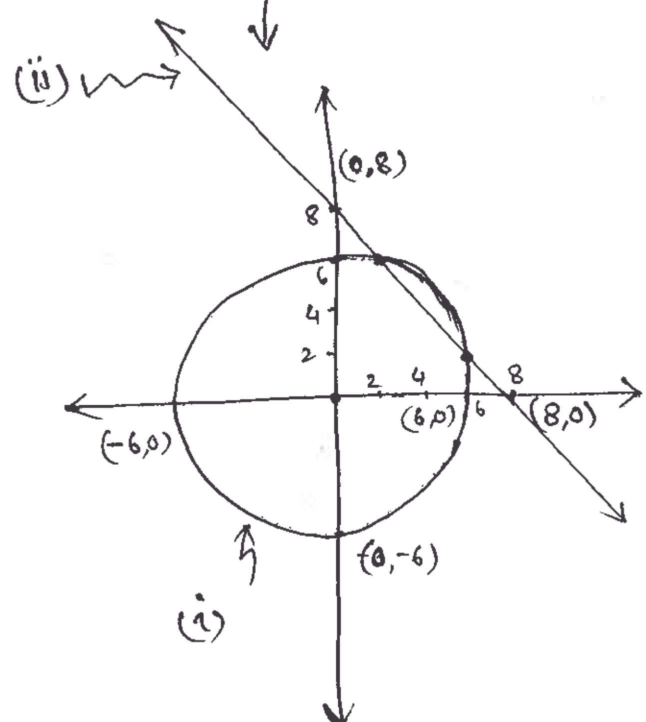


Let us now plot the two equations (i) and (ii) together in a single coordinate plane, which is shown in the graph below.



Notice that the straightline intersects the circle into two points, close to (6,2) and (2,6). Therefore, the equations must have two solutions.

Let us verify them by solving them algebraically. Equation (i) is clearly a nonlinear equation, whereas equation (ii) is linear.



Let us use the method of substitution and substitute the value of y from equation (ii) into equation (i)

$$y = \sqrt{36 - x^2}$$

$$\Rightarrow 8 - x = \sqrt{36 - x^2} \quad (\text{As } y = 8 - x)$$

$\Rightarrow (8-x)^2 = 36 - x^2$ (Squaring both sides)

$\Rightarrow 64 - 16x + x^2 = 36 - x^2$

$\Rightarrow 2x^2 - 16x + 28 = 0 \quad \cdots\cdots (v)$

Equation (v) is a quadratic equation. The solution of which is the following

$$x = \frac{16 \pm \sqrt{(-16)^2 - 4\cdot 2\cdot 28}}{2\cdot 2} = \frac{16 \pm 4\sqrt{2}}{4} = 4 \pm \sqrt{2}$$

Now, substituting values of x in equation (ii), we get

$y = 8 - x$	$y = 8 - x$
$= 8 - (4 + \sqrt{2})$	$= 8 - (4 - \sqrt{2})$
$= 4 + \sqrt{2}$	$= 4 - \sqrt{2}$

$\therefore$ The solutions are $\begin{cases} x = 4+\sqrt{2} \\ y = 4+\sqrt{2} \end{cases}$ $\quad \begin{cases} x = 4-\sqrt{2} \\ y = 4-\sqrt{2} \end{cases}$

The numbers are very close to the values obtained from the graph.

Solving a system of nonlinear equations using elimination

18. $\begin{cases} x^2 + y^2 = 16 & \text{——— (1)} \\ x^2 - 2y = 8 & \text{——— (2)} \end{cases}$

Equation (1) can be compared with the standard form of an equation of a circle $(x-h)^2 + (y-k)^2 = r^2$. Equation (1) is, therefore, a circle of radius 4, centering at origin.

The vertex form of a parabola's equation is generally expressed as $y = a(x-h)^2 + k$, where (h, k) is the vertex of the parabola. The parabola opens up like a regular 'U' if a is positive. If a is negative, then the graph opens downwards like an upside down 'U'.

let us now rewrite equation ② in the following form

$$x^2 - 2y = 8$$

$$\Rightarrow 2y = x^2 - 8$$

$$\Rightarrow y = \frac{1}{2}x^2 - 4$$

$\therefore$ The eqn ② is a parabola, whose graph opens up like a regular 'U', whose vertex is at $(0, -4)$.

As $|a| < 1$, the graph of the parabola widens. This just means that the 'U' shape of parabola streatches out sideways.

let us now use the method of elimination to solve the equations (i) and (ii)

$$\cancel{x^2} + y^2 = 16 \quad \text{—— ①}$$

$$\cancel{x^2} - 2y = 8 \quad \text{—— ②}$$

$$(\cancel{-}) \quad (+) \quad (-)$$

$$y^2 + 2y = 8$$

$$\Rightarrow y^2 + 2y - 8 = 0 \quad \text{— ③}$$

Equation ③ is a quadratic eqn.

$$y = \frac{-2 \pm \sqrt{4 + 32}}{2} = \frac{-2 \pm 6}{2}$$

$\therefore \; y = 2, -4$

Substituting values of y in (2) we, obtain

$$\left.\begin{aligned} x^2 - 2y &= 8 \\ \Rightarrow x^2 - 2\cdot 2 &= 8 \\ \Rightarrow x^2 &= 12 \\ \Rightarrow x &= \pm\sqrt{12} \\ &= \pm 2\sqrt{3} \end{aligned}\right\} \text{when } y = 2 \qquad \left.\begin{aligned} x^2 - 2y &= 8 \\ \Rightarrow x^2 - 2\cdot(-4) &= 8 \\ \Rightarrow x^2 &= 0 \\ \Rightarrow x &= 0 \end{aligned}\right\} \text{when } y = -4$$

$\because$ The equations have three solutions

$$\begin{cases} x = 2\sqrt{3} \\ y = 2 \end{cases} \qquad \begin{cases} x = -2\sqrt{3} \\ y = 2 \end{cases} \qquad \begin{cases} x = 0 \\ y = -4 \end{cases}$$