6.6 Logarithmic and exponential equations

Problems

8. Solve the logarithmic equation. Express irrational solutions in exact form and as a decimal rounded to three decimal places.

$\log_3(3x-1) = 2$

$\Rightarrow (3x-1) = 3^2$

$\Rightarrow 3x - 1 = 9$

$\Rightarrow 3x = 10$

$\Rightarrow x = \frac{10}{3} = 3.333$.

$\log_a x = y$

$\Rightarrow x = a^y$

21. Solve $\log_2(x+7) + \log_2(x+8) = 1$

$\Rightarrow \log_2\{(x+7)(x+8)\} = 1$

$\log_a M + \log_a N$
$= \log_a(MN)$

$\Rightarrow (x+7)(x+8) = 2^1$

$\Rightarrow x^2 + 15x + 56 = 2$

$\Rightarrow x^2 + 15x + 54 = 0$

$\Rightarrow x^2 + 9x + 6x + 54 = 0$

$\Rightarrow x(x+9) + 6(x+9) = 0$

$\Rightarrow (x+9)(x+6) = 0$.

Either, $x = -9$ or, $x = -6$.

39. Solve the exponential equation $5(2^{3x}) = 8$. Express irrational solutions in exact form and as a decimal rounded to three decimal places.

$$5(2^{3x}) = 8$$

$$\Rightarrow 2^{3x} = \frac{8}{5}$$

Taking the natural logarithm on each sides

$$\Rightarrow \ln 2^{3x} = \ln \frac{8}{5}$$

If $M = N$, then $\log_a M = \log_a N$.

$$\Rightarrow 3x \ln 2 = \ln \frac{8}{5}$$

$$\Rightarrow 3x = \frac{\ln \frac{8}{5}}{\ln 2}$$

$$\Rightarrow x = \frac{\ln \frac{8}{5}}{3 \ln 2} = 0.226$$ (Using calculator)

47.

$$\pi^{1-x} = e^x$$

$$\Rightarrow \ln \pi^{1-x} = \ln e^x$$

$$\Rightarrow (1-x) \ln \pi = x \ln e$$

$\ln e = 1$, because $\log_e e = 1$

$$\Rightarrow \ln \pi - x \ln \pi = x$$

$$\Rightarrow x(1 + \ln \pi) = \ln \pi$$

$$\Rightarrow x = \frac{\ln \pi}{1 + \ln \pi} = 0.534$$ (Using calculator)

53.

$$16^x + 4^{x+1} - 3 = 0$$

$$\Rightarrow (4^x)^2 + 4 \cdot 4^x - 3 = 0$$

$16^x = (4^2)^x = 4^{2x} = (4^x)^2$

$4^{x+1} = 4^x \cdot 4^1 = 4 \cdot 4^x$.

Use quadratic formula to solve the above equation

$$\therefore\ 4^x = \frac{-4 \pm \sqrt{4^2+12}}{2} = \frac{-4 \pm \sqrt{28}}{2} = \frac{-4 \pm 2\sqrt{7}}{2}$$

$$= -2 \pm \sqrt{7}$$

$\therefore$ Either, $4^x = -2+\sqrt{7}$

$\Rightarrow \ln 4^x = \ln(-2+\sqrt{7})$

$\Rightarrow x \ln 4 = \ln(-2+\sqrt{7})$

$\Rightarrow x = \dfrac{\ln(-2+\sqrt{7})}{\ln 4}$

$= -0.315$ (Using calculator)

or, $4^x = -2-\sqrt{7}$

$\Rightarrow \log_4(-2-\sqrt{7}) = x$ | $\log_a x = y \Rightarrow x = a^y$

$\Rightarrow x = 1.108 + i \cdot 2.267$ (Complex number) (Using calculator)

$\therefore$ The only real solution is $x = -0.315$.

6.7 Compound interest

We know the formula for the simple interest

$$I = Prt,$$

where, $\begin{cases} P = \text{Principal} \\ r = \text{Rate of interest} \\ t = \text{time}. \end{cases}$

Once you know the above three quantities, you can calculate the amount of interest that you have to pay. However, in this chapter our aim is to calculate the compound interest. We will derive a formula for that. Before doing that let us solve a problem first.

Problem

7. Find the amount that results from the following investment. \$100 invested at 4% compounded quarterly after a period of 2 years.

The principal P is \$100, rate of interest is $4\% = 0.04$. After the first quarter of the first year, the time t is $\frac{1}{4}$ year. The interest earned after the first quarter is

$$I_1 = P.r.t = 100 \times 0.04 \times \frac{1}{4} = \$1$$

The new principal is $P + I_1 = \$100 + \$1 = \$101$. At the end of the second quarter, the interest is

$$I_2 = P.r.t = \$101 \times 0.04 \times \frac{1}{4} = \$1.01.$$

At the end of third quarter, the interest on the new principal of $\$101 + \$1.01 = \$102.01$ is

$$I_3 = P.r.t = 102.01 \times 0.04 \times \frac{1}{4} = \$1.02$$

After the 4th quarter, the interest is

$$I_4 = P.r.t = 103.03 \times 0.04 \times \frac{1}{4} = \$1.03.$$

After one year, the total amount $= \$103.03 + \$1.03$

$$= \$104.06$$

After the 1st quarter of the 2nd year, the interest is

$$I_5 = 104.06 \times 0.04 \times \frac{1}{4} = \$1.04$$

Similarly

$$I_6 = 105.1 \times 0.04 \times \frac{1}{4} = \$1.05$$

$$I_7 = 106.15 \times 0.04 \times \frac{1}{4} = \$1.06$$

$$I_8 = 107.21 \times 0.04 \times \frac{1}{4} = \$1.07$$

$\therefore$ The final amount after the end of 2 years is $= 107.21 + 1.07 = 108.28$

Now let us look at the pattern of the previous problem. To fix our ideas, let P represents the principal to be invested at a per annum interest rate r that is compounded n times per year, so the time of each compounding period is $\frac{1}{n}$ years. The interest earned after each compounding period is given by

$$I = P \cdot r \cdot t = P \cdot r \cdot \frac{1}{n} = P \cdot \frac{r}{n}$$

The amount A after one compounding period is

$$A = P + P \cdot \frac{r}{n} = P\left(1+\frac{r}{n}\right)$$

After two compounding periods, the amount A, based on the new principal $P\left(1+\frac{r}{n}\right)$ is

$$A = \underbrace{P\left(1+\frac{r}{n}\right)}_{\text{New principal}} + \underbrace{P\cdot\left(1+\frac{r}{n}\right)\left(\frac{r}{n}\right)}_{\text{Interest on new principal}} = P\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right) = P\cdot\left(1+\frac{r}{n}\right)^2$$

After three compounding periods, the amount A is

$$A = P\left(1+\frac{r}{n}\right)^2 + P\left(1+\frac{r}{n}\right)^2\left(\frac{r}{n}\right) = P\left(1+\frac{r}{n}\right)^2\left(1+\frac{r}{n}\right) = P\left(1+\frac{r}{n}\right)^3.$$

Continuing this way, after n compounding periods (1 year), the amount A is

$$A = P\left(1+\frac{r}{n}\right)^n$$

Because t years will contain n·t compounding periods, after t years we have

$$\boxed{A = P\cdot\left(1+\frac{r}{n}\right)^{nt}}$$

P = Principal
r = Annual interest rate
n = times compounded/year
t = Number of years.
A = total amount that one obtains at the end

Problem

8. Find the amount that results from the following investment. \$50 invested at 6% compounded monthly after a period of 3 years.

We will now use the formula, that we have just derived

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$= 50\left(1+\frac{0.06}{12}\right)^{12\times 3}$$

$$= 59.83$$

P = Principal = \$50
r = 6% = 0.06
n = 12
t = 3 years.

Continuous compounding

One obvious fact that we have observed is that we obtain higher amount of money when we invest it to the scheme of compound interest, rather than the scheme of simple interest. Now the question is what would happen to the amount after 1 year if the number of times that the interest is compounded were increased without bound? Let us find the answer. We know the amount after 1 year is

$$A = P\left(1+\frac{r}{n}\right)^{n}$$

Rewrite the expression as follows:

$$A = P\left(1+\frac{r}{n}\right)^{n} = P\left(1+\frac{1}{\frac{n}{r}}\right)^{n} = P\left[\left(1+\frac{1}{\frac{n}{r}}\right)^{n/r}\right]^{r} = P\left[\left(1+\frac{1}{h}\right)^{h}\right]^{r},$$

where $h=\frac{n}{r}$. Now suppose that the number n of times that the interest compounded per year gets larger and larger; i.e. suppose that $n\to\infty$. Then $h=\frac{n}{r}\to\infty$, and the expression in brackets in the previous equation equals e. If you remember from the previous class -

$$e = \lim_{h\to\infty}\left(1+\frac{1}{h}\right)^{h}$$

$\therefore A = Pe^{r}$

So the amount A after t years due to principal P invested at an annual interest rate r compounded continuously is

$$\boxed{A = Pe^{rt}}$$

Problem

14. Find the amount that results from the following investment. \$400 invested at 7% compounded continuously after a period of 3 years.

$$A = 400 \cdot e^{0.07 \times 3} = 400 \times 1.23 = \$\,493.471$$

16. Find the principal needed now to get the following amount. i.e. find the present value.
To get \$75 after 3 years at 8% compounded quarterly.

When people in finance speak of the "time value of money", they are usually referring to the present value of money. The present value of A dollars to be received at a future date is the principal that you would need to invest now so that it will grow to A dollars in the specified time period.

We will use the compound interest formula

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$\Rightarrow P = \frac{A}{\left(1+\frac{r}{n}\right)^{nt}} = A\left(1+\frac{r}{n}\right)^{-nt}$$

$$= 75\left(1+\frac{0.08}{4}\right)^{-4\times 3}$$

$$= 59.14$$

So, one needs the principal of \$59.14 to get \$75 after 3 years.

24. Find the effective rate of interest for 6% compounded monthly.

The effective rate of interest is the equivalent annual simple interest rate that would yield the same amount as compounding n times per year, or continuosly of the 1 year.

The effective rate of interest of an investment earning an annual interest rate r is given by

$$r_e = \left(1+\frac{r}{n}\right)^n - 1 \quad \text{for compounding } n \text{ times per year.}$$

$$r_e = e^r - 1 \quad \text{for continuous compounding.}$$

$$\therefore \ r_e = \left(1+\frac{0.06}{12}\right)^{12} - 1$$

$$= 0.0616778$$

$$= 6.168\ \%$$

Ah.

27. Determine the rate that represents the better deal.

6% compounded quarterly or $6\frac{1}{4}$% compounded annually.

To test the better deal, one needs to calculate the effective interest rate in each cases.

1st case

$$r_e = \left(1+\frac{r}{n}\right)^n - 1$$

$$= \left(1+\frac{0.06}{4}\right)^4 - 1$$

$$= 0.0613636$$

$$= 6.136\%$$

2nd case

$$r_e = \left(1+\frac{r}{n}\right)^n - 1$$

$$= \left(1+\frac{0.0625}{1}\right)^1 - 1$$

$$= 0.0625$$

$$= 6.25\%$$

Clearly the 2nd deal is the best deal.

6.8 Exponential growth and decay models

Many naturally occurred phenomena have been found to follow the law that an amount A varies with time t according to the function

$$A(t) = A_0 e^{Kt}.$$

Here, A_0 is the original amount $(t=0)$ and $K \neq 0$ is a constant. If $K > 0$, then the above equation states that the amount A is increasing over time, whereas if $K < 0$, the amount A is decreasing over time. In either cases, when an amount A varies over time according to the above equation, it is said to follow the exponential law or the law of uninhibited growth $(K>0)$ or decay $(K<0)$.

Problems

4. Radioactive decay: Iodine is a radioactive material that decay according to the function $A(t) = A_0 e^{-0.087t}$ where A_0 is the initial amount present and A is the amount present at time t (in days). Assume that a scientist has a sample of 100 grams of iodine 131.
   (a) what is the decay rate of iodine 131?
   (b) How much iodine 131 is left after 9 days?
   (c) When will 70 grams of iodine 131 be left?
   (d) what is the half life of iodine 131?

(a) $$A(t) = 100\, e^{-0.087t}$$

Compare the above equation with $A(t) = A_0 e^{-Kt}$. The value of K, 0.087 indicates a decay rate of 8.7 %.

(b) $A(9) = 100\, e^{-0.087 \times 9}$

$= 45.7033$ (Using calculator)

$\therefore$ 45.70 grams of iodine 131 is left after 9 days.

(c) $A(t) = 100\, e^{-0.087 t}$

$\Rightarrow 70 = 100\, e^{-0.087 t}$

$\Rightarrow e^{0.087 t} = \frac{100}{70}$

$\Rightarrow \ln e^{0.087 t} = \ln\left(\frac{10}{7}\right)$

$\Rightarrow 0.087\, t \ln_e e = \ln\left(\frac{10}{7}\right)$

$\Rightarrow 0.087\, t = \ln 10 - \ln 7$

$\Rightarrow t = \frac{\ln 10 - \ln 7}{0.087}$

$= 4.0997$

Therefor, approximately after 4 days, 70 grams of iodine 131 will be left.

(d) Half life of a radioactive material is the time in which the material decays to half of its initial value.

$\therefore A(t) = 100\, e^{-0.087 t}$

$\Rightarrow 50 = 100\, e^{-0.087 t}$

$\Rightarrow e^{0.087 t} = \frac{100}{50}$

$\Rightarrow 0.087 t = \ln 2$

$\therefore t = \frac{\ln 2}{0.087} = 7.967$

$\therefore$ Half life of iodine 131 is approximately 8 days.

7. Population growth : The population of a southern city follows the exponential law.
(a) If N is the population of the city and t is the time in years, express N as a function of t.
(b) If the population doubled in size over an 18-month period and the current population is 10,000, what will the population be 2 years from now?

(a) $$N(t) = N e^{Kt}$$

(b) $$N(t) = 2N \quad (\text{Population is doubled})$$
$$t = 1.5 \quad (18 \text{ month} = 1.5 \text{ years})$$

$$\because \; 2\cancel{N} = \cancel{N} e^{1.5K}$$
$$\Rightarrow \ln 2 = \ln e^{1.5K}$$
$$\Rightarrow \ln 2 = 1.5K$$
$$\Rightarrow K = \frac{\ln 2}{1.5} = 0.462 = 46.2\%$$

$\therefore$ The growth rate is 46.2%.

$$\therefore \; N(t) = N e^{Kt}$$
$$= 10000 \, e^{0.462 \times 2}$$
$$= 25193.5 \approx 25193$$

$\therefore$ The population will be 25193 in 2 years from now.