6·3 Exponential function :

Laws of exponents

If s, t, a and b are real numbers with $a>0$ and $b>0$, then

(i) $a^s \cdot a^t = a^{s+t}$ (ii) $(a^s)^t = a^{st}$

(iii) $(ab)^s = a^s \cdot b^s$ (iv) $1^s = 1$

(v) $a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s$ (vi) $a^0 = 1$.

Exponential growth

The general form of an exponential function is

$$f(x) = c \cdot a^x$$

where a is a real positive number $(a>0)$; $a \neq 1$ and $c \neq 0$ is another real number.

Example

(i) $f(x) = 10 \cdot 2^x$

Let us see few terms from the begining -

$f(0) = 10 \cdot 2^0 = 10 \cdot 1 = 10$

$f(1) = 10 \cdot 2^1 = 10 \cdot 2 = 20$

$f(2) = 10 \cdot 2^2 = 10 \cdot 4 = 40$

$f(3) = 10 \cdot 2^3 = 10 \cdot 8 = 80$

$f(4) = 10 \cdot 2^4 = 10 \cdot 16 = 160$

Notic that the function has two following properties -

1. The value of $f(x)$ doubles with every 1-unit increase in the independent variable x.
2. The value of $f(x)$ at $x=0$ is a real positive number. $f(0) = 10$

The function $f(x) = 10 \cdot 2^x$ is, therefore, an exponential function of the growth factor 2.

(ii) Let us consider another example:

$$f(x) = \left(\frac{1}{2}\right)^x + 10$$

$$\Rightarrow f(0) = \left(\frac{1}{2}\right)^0 + 10 = 1 + 10 = 11 = (+ve) \text{ real number.}$$

$$\therefore f(1) = \left(\frac{1}{2}\right)^1 + 10 = \frac{1}{2} + 10 = 10\frac{1}{2}$$

$$f(2) = \left(\frac{1}{2}\right)^2 + 10 = \frac{1}{4} + 10 = 10\frac{1}{4}$$

So, you see the function is decreasing with the increase of x, that is why the function $f(x) = \left(\frac{1}{2}\right)^x + 10$ is an exponentially decaying function. While the function we considered in the previous example, is an exponentially increasing function.

(iii) $f(x) = 10 \cdot 3^{x-3}$

Note that, the above function is a shifted function along the positive x-axis, where the original function is $10 \cdot 3^x$. That means, any function which is transformed (vertically, horizontally, reflected and so on) function of an exponential function, is also an exponential function.

Identifying linear or exponential function

Let us go back again to the example number (i). Notice that the ratio of two consecutive outputs is constant for 1-unit increase of the input. The ratio equals to the constant 2, which is the base of the exponential function. In other words,

$$\frac{f(1)}{f(0)} = \frac{20}{10} = 2 \ ; \quad \frac{f(2)}{f(1)} = \frac{40}{20} = 2 \ ; \quad \frac{f(3)}{f(2)} = \frac{80}{40} = 2 .$$

So, an exponential function can be identified in general by computing the ratio of the consecutive outputs. If the outcome is a constant, then the function is surely an exponential function.

$$\frac{f(x+1)}{f(x)} = a$$

Whereas, the growth of the linear function, as discussed in one of the previous lectures, can be identified by computing the average rate of change of y with respect to x. If the average rate of change is a constant, then the function is linear.

<u>Graph exponential function</u>

let us consider an exponential function $f(x) = 3^x$. At first, we make a table of the values of x versus $f(x)$.

x	-10	-3	-2	-1	0	1	2	3	10
$f(x)$	0.000017	0.037	0.111	0.333	1	3	9	27	59049

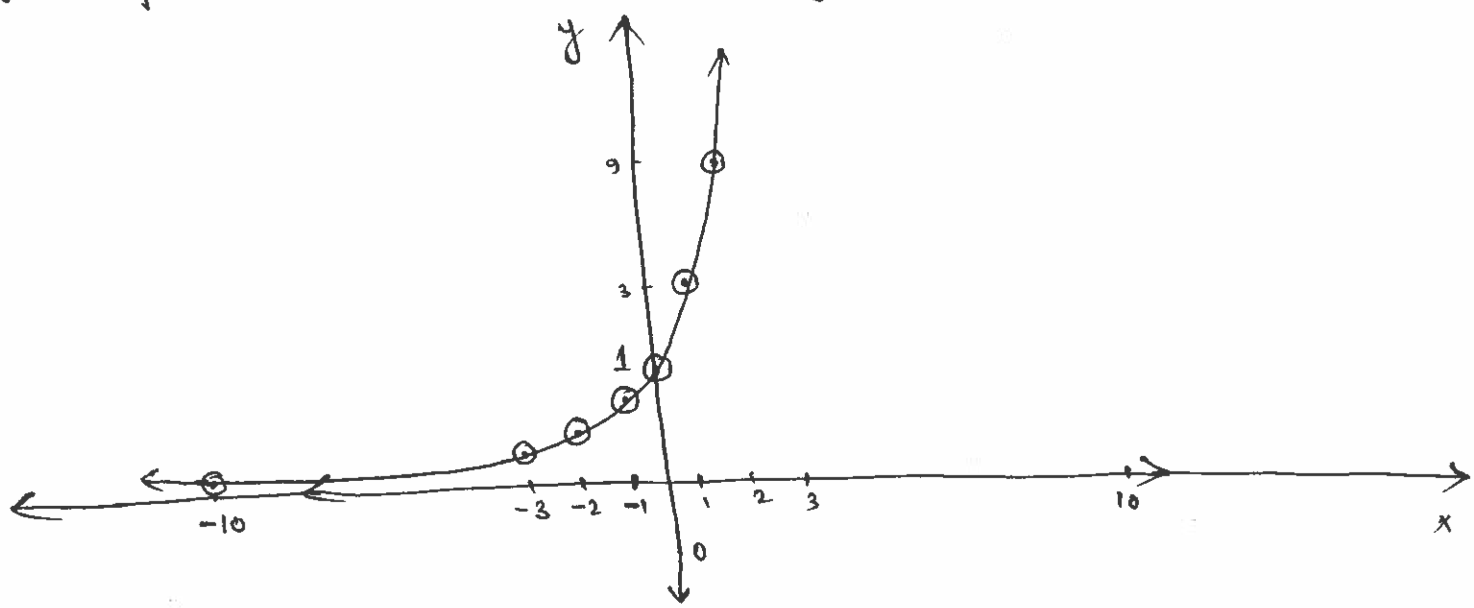
<u>Domain</u> of the function : All real numbers, as the fun. is allowed for any real number x.

<u>Range</u> : Since $3^x > 0$ for all x, the range of $f(x)$ is $(0, \infty)$. From this, we conclude that the graph has <u>no x-intercepts</u> and the graph will lie above the x-axis.

<u>y-intercept</u> is found by putting $(x=0)$ in the function, which is 1. So, the y-intercept is at 1.

The table suggests that as <u>$x \to -\infty$</u>, the values of $f(x) = 3^x$ get closer and closer to <u>zero</u>. $\lim\limits_{x \to -\infty} f(x) = 0$

The horizontal asymptote of the graph is therefore the x-axis. The table also suggests that as $x \to \infty$, the values of $f(x) = 3^x$ grow very quickly, causing the graph of $f(x) = 3^x$ to rise very rapidly. Hence, it is clear that function is an increasing exponential function.



Define the number e

As we shall see shortly, many problems that occur in nature require the use of an exponential function, whose base is a certain irrational number, symbolized by the letter 'e'. The number is defined as the number that the expression

$\left(1+\frac{1}{n}\right)^n$ approaches as $n \to \infty$. In the notation of calculus, this is expressed as –

$$e = \lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n \approx 2{\cdot}71828$$

Problems

43. Use transformations to graph each function. Determine the domain, range and horizontal asymptote of each function.

$f(x) = 3^{x-1}$

In the previous problem, we saw the graph of the function $f(x) = 3^x$. We will use the transformation rule again to transform the graph to 3^{x-1}. clearly, the graph shifts 1 unit to the right along the x-axis.

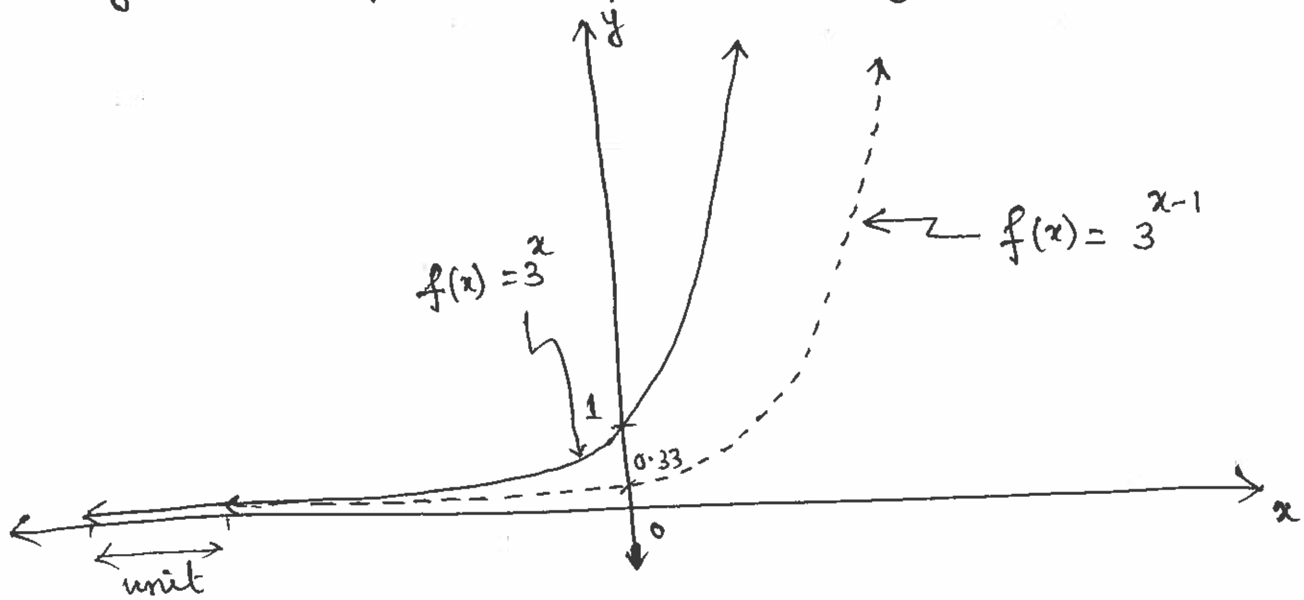
~~Range~~ Domain of $f(x) = 3^{x-1}$: All real numbers.

~~Domain~~ Range of $f(x) = 3^{x-1}$: $(0, \infty)$

Horizontal asymptote : $y = 0$ line. (x-axis)

x-intercept : No x-intercepts

y-intercept : $f(0) = 3^{-1} = \frac{1}{3} = 0.333$.



$\underline{50}$. $f(x) = 1 + 2^{x+3}$

The nature of the graph of $f(x) = 2^x$ is quite similar to that of the graph of $f(x) = 3^x$. Transformation rules suggest that the graph will be shifted 3 units to the (-ve) x axis first and then 1 unit along the (+ve) y-axis.

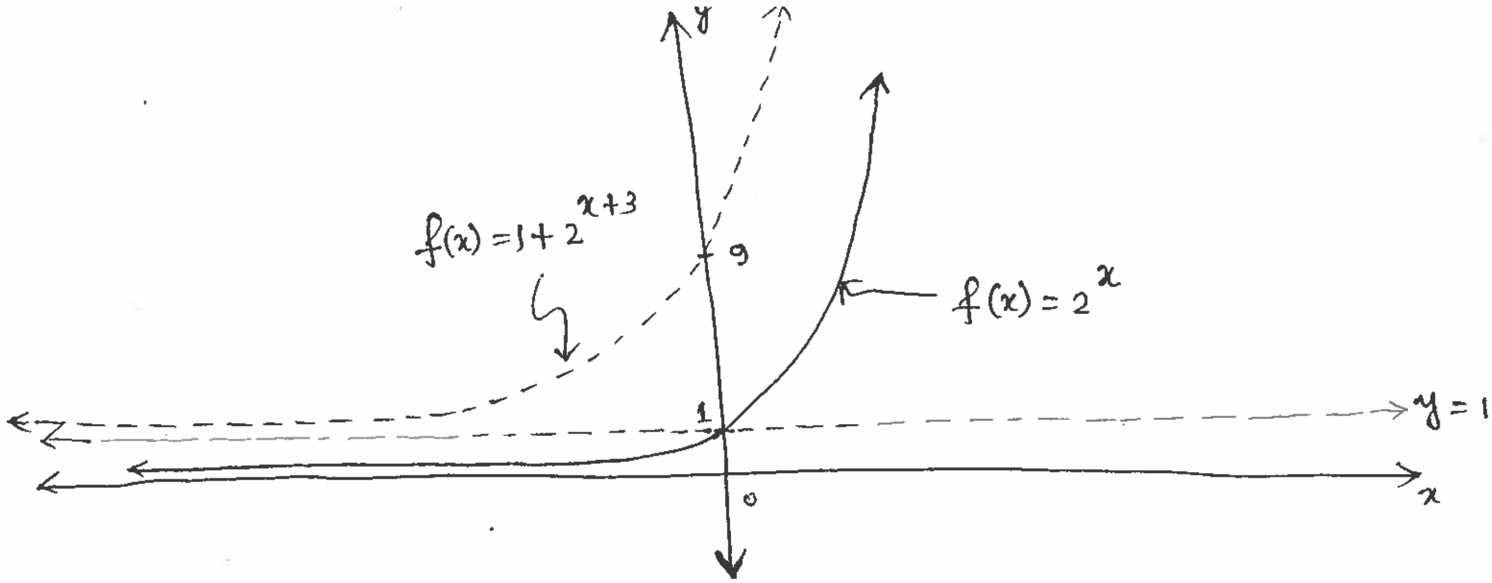
Domain ~~Range~~ of $f(x) = 1 + 2^{x+3}$: All real numbers.

Range of $f(x) = 1 + 2^{x+3}$: $(1, \infty)$

Horizontal asymptote : $y = 1$ line

x-intercept : No x-intercepts

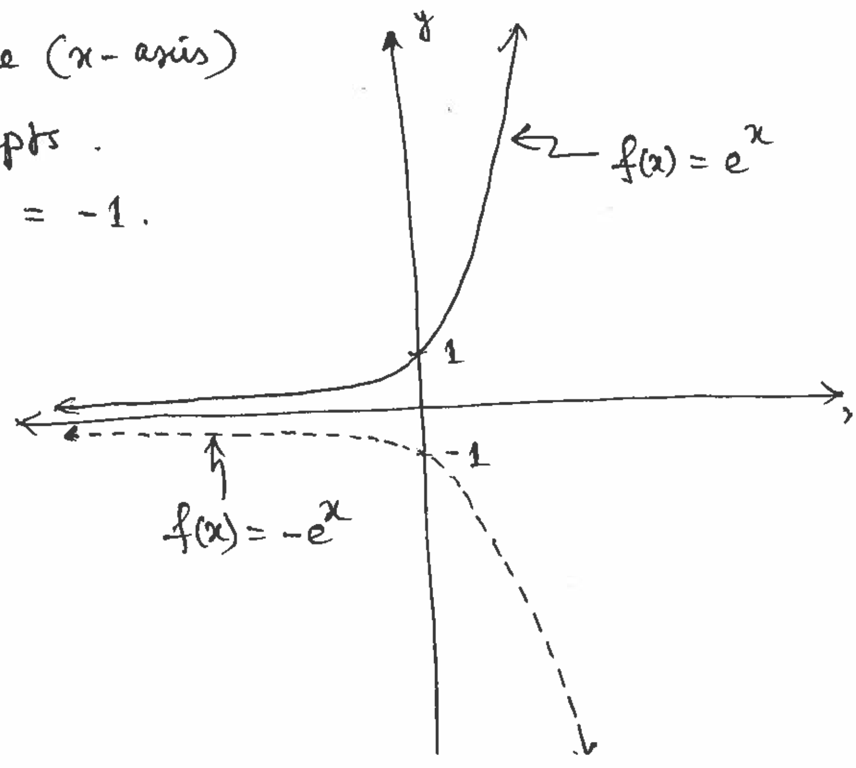
y-intercept : $f(0) = 1 + 2^3 = 9$.



54. Begin with the graph of $y = e^x$ and use transformations to graph the function. Determine the domain, range and horizontal asymptote.
$$f(x) = -e^x.$$

The graph of $f(x) = e^x$ is very similar to the graphs of 2^x and 3^x, because the value of $e \approx 2{\cdot}71828$. One then needs to take the reflection about the x-axis to obtain the graph of $f(x) = -e^x$.

Domain of $(-e^x)$: All real numbers.
Range of $(-e^{-x})$: $(-\infty, 0)$
Horizontal asymptote : $y = 0$ line (x-axis)
x-intercept : No x-intercepts.
y-intercept : $f(0) = -e^0 = -1$.



63. Solve $2^{-x} = 16$

$$2^{-x} = 2^4$$
$$\Rightarrow -x = 4$$
$$\Rightarrow x = -4$$

68. Solve $5^{x+3} = \frac{1}{5}$

$$5^{x+3} = \frac{1}{5}$$
$$\Rightarrow 5^{x+3} = (5)^{-1}$$
$$\Rightarrow x+3 = -1$$
$$\Rightarrow x = -4$$

76. Solve $9^{2x} \cdot 27^{x^2} = 3^{-1}$

$$9^{2x} \cdot 27^{x^2} = 3^{-1}$$
$$\Rightarrow (3^2)^{2x} \cdot (3^3)^{x^2} = 3^{-1}$$
$$\Rightarrow 3^{4x} \cdot 3^{3x^2} = 3^{-1}$$
$$\Rightarrow 3^{4x+3x^2} = 3^{-1}$$
$$\Rightarrow 4x+3x^2 = -1$$
$$\Rightarrow 3x^2+4x-1$$

$$\therefore x = \frac{-4 \pm \sqrt{16+4\cdot 3\cdot 1}}{2\cdot 3}$$
$$= \frac{-4 \pm \sqrt{28}}{6}$$
$$= \frac{-4 \pm 2\sqrt{7}}{6}$$
$$= \frac{-2 \pm \sqrt{7}}{3}.$$

78. Solve $e^{3x} = e^{2-x}$

$$\therefore 3x = 2-x$$
$$\Rightarrow 4x = 2$$
$$\Rightarrow x = \frac{1}{2}$$

6·4 Logarithmic function

Recall that a one-to-one function $y = f(x)$ has an inverse function that is defined by the equation $x = f(y)$. In particular the exponential function $y = f(x) = a^x$; where $a > 0$ and $a \neq 1$, is one-to-one and hence has an inverse function that is defined by the equation

$$x = a^y \quad ; \quad a > 0 ; \quad a \neq 1 .$$

This inverse function is so important that it is given a name, the logarithmic function. The logarithmic function to the base a, where $a > 0$ and $a \neq 1$ is denoted by $y = \log_a x$ (read as "y is the logarithm to the base a of x") and is defined by

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

Examples

1. If $y = \log_5 x$, then $x = 5^y$
2. If $-1 = \log_5(\frac{1}{5})$, then $\frac{1}{5} = 5^{-1}$
3. If $e^b = 9$, then $b = \log_e 9$

Problems

28. Find the exact value of the logarithm, without using calculator.

$y = \log_3 (1/9)$

$\Rightarrow y = \log_3\left(\frac{1}{3^2}\right) = \log_3 (3^{-2})$

$\therefore 3^{-2} = 3^y$

$\therefore y = -2.$

| $y = \log_a x$ is equivalent to $x = a^y$

33. Find the exact value of each logarithm

$$y = \log_{\sqrt{2}} 4 \qquad\qquad y = \log_a x \Leftrightarrow x = a^y$$

$$= \log_{\sqrt{2}} 2^2$$

$$\therefore 2^2 = \sqrt{2}^{\,y}$$

$$\Rightarrow 2^2 = \left(2^{1/2}\right)^y$$

$$\Rightarrow 2^2 = 2^{y/2}$$

$$\therefore \frac{y}{2} = 2$$

$$\Rightarrow y = 4$$

36. Find the exact value of the logarithm $\ln e^3$.

'ln' is the logarithm to the base 'e'. ln is called the natural logarithm, because the natural number e is in the base.

$$y = \ln e^3$$

$$= \log_e e^3 \qquad\qquad y = \log_a x \Leftrightarrow x = a^y$$

$$\therefore e^3 = e^y$$

$$\Rightarrow y = 3$$

Domain of the logarithmic function

The logarithmic function $y = \log_a x$ has been defined as the inverse of the exponential function $y = a^x$. That is if $f(x) = a^x$, then $f^{-1}(x) = \log_a x$. Based on the discussion given in section 6.2 on inverse functions for a function f and its inverse f^{-1}, we have

Domain of f^{-1} = Range of f and Range of f^{-1} = Domain of f.

consequently, it follows that

Domain of the logarithmic function = Range of the exponential function $= (0, \infty)$

Range of the logarithmic function = Domain of the exponential function $= (-\infty, \infty)$

Problems

37. Find the domain of the function $f(x) = \ln(x-3)$

Domain of the function $f(x)$ consists of all x for which

$x - 3 > 0$

$\Rightarrow x > 3$.

In the interval notation the domain of $f(x)$ is $(3, \infty)$

44. Find the domain of the function $f(x) = \ln\left(\frac{1}{x-5}\right)$

$\frac{1}{x-5} > 0$

$\Rightarrow x-5 > 0$

$\Rightarrow x > 5$

If $\frac{1}{x} > 0$, then $x > 0$

If $x > 0$, then $\frac{1}{x} > 0$

$\therefore$ The domain of the function is $(5, \infty)$

82. Use the function $f(x) = 2 - \log_3(x+1)$ to

(a) Find the domain of $f(x)$ (b) Graph f

(c) From the graph, determine the range and any asymptotes of f.

(d) find f^{-1}, the inverse of f

(e) Find the domain and range of f^{-1} (f) Graph f^{-1}.

(a) Domain of $f(x)$ is all x for which

$x + 1 > 0$

$\Rightarrow x > -1$ $\therefore$ Domain: $(-1, \infty)$

(b) Recall that a logarithmic function $y = \log_a x$ is equivalent to $x = a^y$. The inverse of $x = a^y$ is obtained by interchanging x and y, which is equal to $y = a^x$.

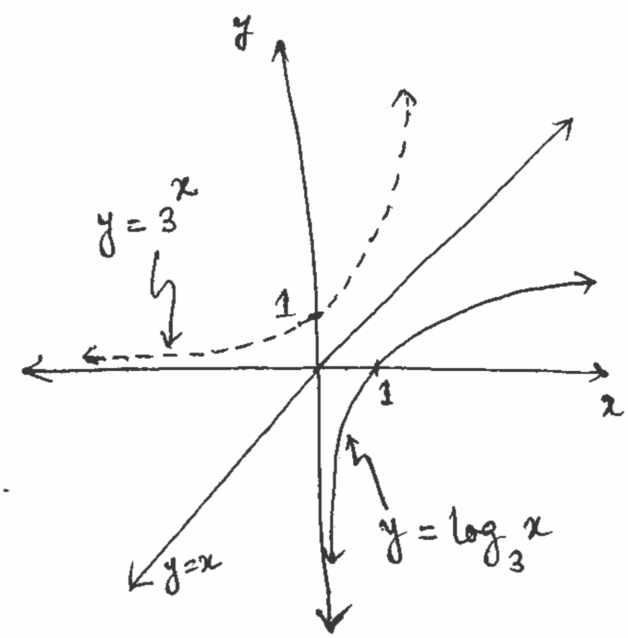
$\therefore$ The inverse of $y = \log_a x$ is $y = a^x$.

You also know that a function and its inverse are symmetric with respect to the $y = x$ line. Therefore, the graph of the logarithmic function is obtained by graphing its inverse function $y = a^x$.

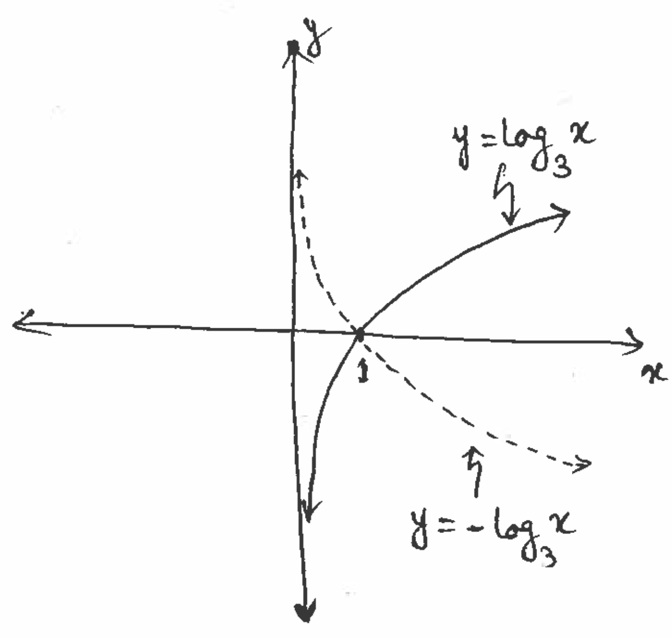
To graph the given function $f(x) = 2 - \log_3(x+1)$, we begin with the graph of $y = \log_3 x$ and then use the transformations.

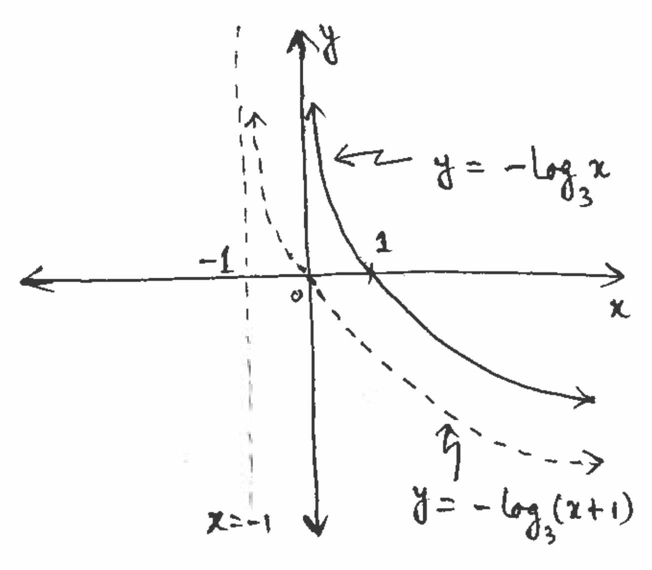
$$y = \log_3 x \iff x = 3^y$$

Inverse of $y = \log_3 x$ is $y = 3^x$.

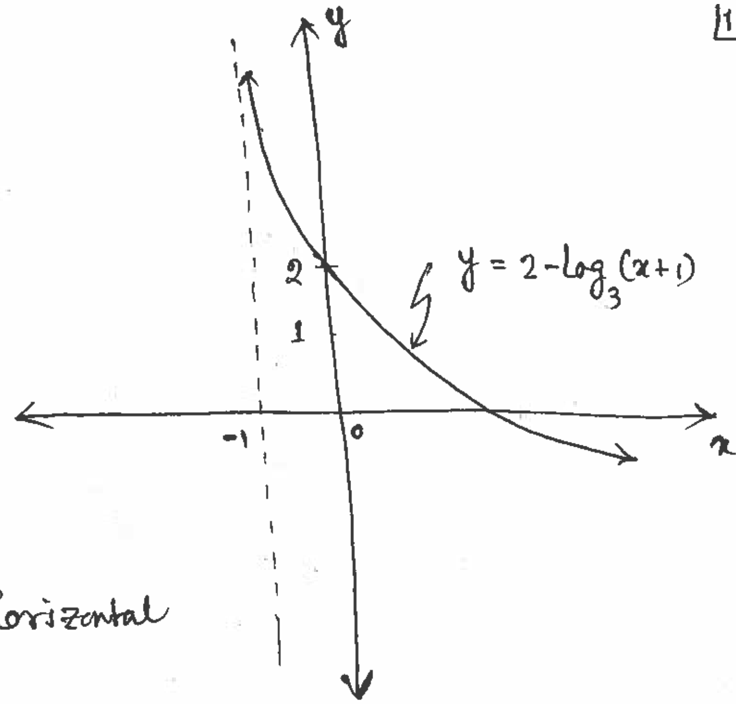


Let us now draw the graph of $f_1(x) = -\log_3(x+1)$. We need to take the reflection about x-axis and then shift the graph by 1 unit along (-ve) x-axis.





Finally, we plot $f_2(x) = 2 - \log_3(x+1)$, by shifting the graph by 2 units along (+ve) y-axis.



(c) It is clear from the graph that the range of the function is $(-\infty, \infty)$.

The vertical asymptote of the function is $x=-1$. There is no horizontal asymptote of the graph.

(d) We now calculate the inverse of the function $y = 2-\log_3(x+1)$

The inverse is $x = 2-\log_3(y+1)$

$$\Rightarrow x-2 = -\log_3(y+1)$$
$$\Rightarrow 2-x = \log_3(y+1)$$
$$\Rightarrow 3^{2-x} = y+1$$
$$\Rightarrow y = 3^{2-x}-1$$

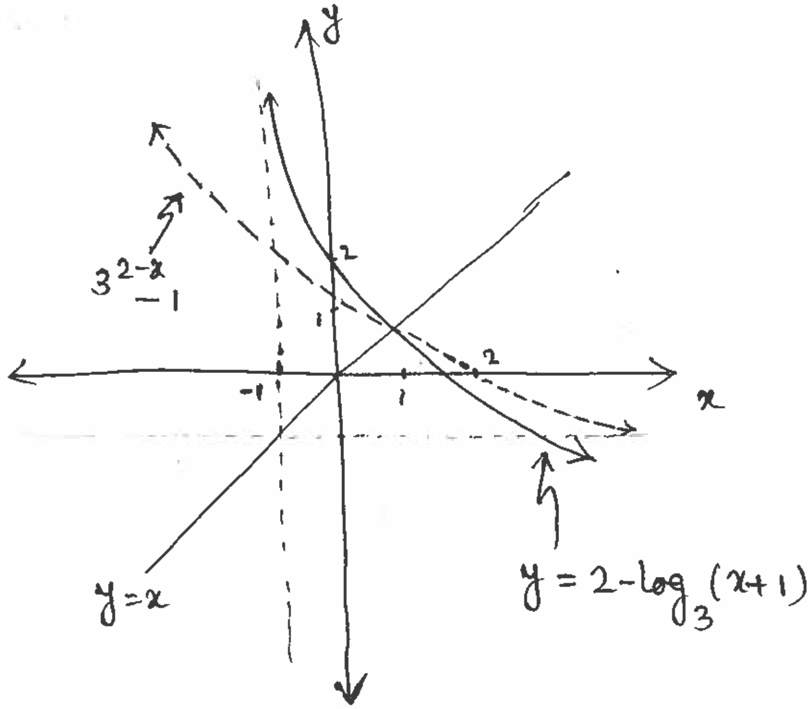
$y = \log_a x$
$\Rightarrow x = a^y$

$$\therefore f^{-1}(x) = 3^{2-x}-1$$

(e) Domain of f^{-1} : Range of f $= (-\infty, \infty)$

Range of f^{-1} : Domain of f $= (-1, \infty)$

(f) The graph of $f^{-1}(x)$ is ~~shifted 1 unit along the (-ve) y-axis~~ obtained by taking a reflection of $f(x)$ with respect to $y=x$ line. One can also use the transformation rule to plot the graph.



90. Solve $\log_3(3x-2) = 2$

$$\therefore\ 3x-2 = 3^2$$

$$\Rightarrow 3x = 9+2$$

$$\Rightarrow x = \frac{11}{3}.$$

$$y = \log_a x$$

$$\Rightarrow x = a^y$$

92. Solve $\log_x\left(\frac{1}{8}\right) = 3$

$$\therefore\ \frac{1}{8} = x^3 \quad \Rightarrow \quad x = \sqrt[3]{\frac{1}{8}}$$

$$\cancel{\Rightarrow \frac{1}{2^3} = x^3}$$

$$\cancel{\Rightarrow 2^{-3} = x^3}$$

$$= \sqrt[3]{\left(\frac{1}{2}\right)^3}$$

$$= \frac{1}{2}$$

94. Solve $\ln e^{-2x} = 8$

$$\therefore\ \ln e^{-2x} = 8$$

$$\therefore\ e^{-2x} = e^8$$

$$\therefore\ -2x = 8 \quad \Rightarrow \quad x = -4$$

6.5 Properties of logarithms

(i) let us first evaluate a simple expression

$$y = \log_a 1$$

$$\Rightarrow a^y = 1$$

$$\Rightarrow a^y = a^0$$

$$\Rightarrow y = 0$$

$\therefore \boxed{\log_a 1 = 0}$ ← This is an important property of a logarithm function, that one should remember.

ii) let us also evaluate the following -

$$y = \log_a a$$

$$\Rightarrow a^y = a$$

$$\Rightarrow a^y = a^1$$

$$\Rightarrow y = 1$$

$\therefore \boxed{\log_a a = 1}$ ← Another interesting property.

(iii) We know $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

Now, use $f(x) = a^x$ and $f^{-1}(x) = \log_a x$ to calculate $f(f^{-1}(x))$

$$f(f^{-1}(x)) = f(\log_a x)$$

$$= a^{\log_a x}$$

$$= x \quad (\text{known to us})$$

$f(x) = a^x$

$f(\log_a x) = a^{\log_a x}$

$\therefore \cancel{\boxed{\log_a a^x = x}}$ ~~for $x>0$~~.

$\therefore \boxed{a^{\log_a x} = x}$ for $x > 0$.

iv) We now compute $f^{-1}(f(x))$, where $f(x)$ and $f^{-1}(x)$ remains same.

$$f^{-1}(f(x)) = f^{-1}(a^x) = \log_a a^x = x$$

$f^{-1}(x) = \log_a x$
$f^{-1}(a^x) = \log_a a^x$

$\therefore \boxed{\log_a a^x = x}$

(v) let us consider $A = \log_a M$ and $B = \log_a N$

$$A = \log_a M \Rightarrow M = a^A$$

$$B = \log_a N \Rightarrow N = a^B$$

Now, let us evaluate

$$\log_a(MN) = \log_a(a^A a^B) = \log_a(a^{A+B})$$

$$= A + B$$ From property (iv)

$$= \log_a M + \log_a N$$

$\therefore \boxed{\log_a(MN) = \log_a M + \log_a N}$

(vi) $\log_a\left(\frac{M}{N}\right) = \log_a\left(\frac{a^A}{a^B}\right) = \log_a(a^{A-B})$

$$= A - B = \log_a M - \log_a N$$

$\therefore \boxed{\log_a(M/N) = \log_a M - \log_a N}$

(vii) $\log_a M^r = \log_a (a^A)^r = \log_a (a^{rA})$

$= rA = r\log_a M$

$\therefore \boxed{\log_a M^r = r\log_a M}$

(viii)

From property (iii) we know

$a^{\log_a x} = x$

We replace a by e

$\therefore e^{\log_e x} = x \Rightarrow e^{\ln x} = x$

Now let $x = a^z$

$\therefore e^{\ln a^z} = e^{z\ln a}$ (Property vii)

$= x$

$= a^z$

$\therefore \boxed{e^{z\ln a} = a^z}$

(ix) Consider $y = \log_a M$ and $z = \log_a N$

$\Rightarrow M = a^y$ $\Rightarrow N = a^z$

where M and N are real positive numbers.

If $M = N$

$\Rightarrow a^y = a^z$

$\Rightarrow y = z$

$\therefore \log_a M = \log_a N$

$\therefore$ $\boxed{\text{If } M=N\text{, then } \log_a M = \log_a N}$

Also $\boxed{\text{If } \log_a M = \log_a N\text{, then } M = N}$

(x) let $y = \log_a M$

$\Rightarrow a^y = M$

$\Rightarrow \log_b a^y = \log_b M$ (Property ix)

$\Rightarrow y \log_b a = \log_b M$ (Property vii)

$\Rightarrow y = \dfrac{\log_b M}{\log_b a}$

$\therefore \boxed{\log_a M = \dfrac{\log_b M}{\log_b a}}$

Problems

22. Use properties of logarithms to find the exact value. Do not use calculator.

$\log_8 16 - \log_8 2$

$= \log_8 (8 \cdot 2) - \log_8 2$ | $\log_a (MN) = \log_a M + \log_a N$ (Property v)

$= \log_8 8 + \cancel{\log_8 2} - \cancel{\log_8 2}$

$= \log_8 8$ | $\log_a a = 1$ (Property ii)

$= 1$

36. Suppose that $\ln 2 = a$ and $\ln 3 = b$. Use properties of logarithm to write each logarithm in terms of a and b.

$\ln \sqrt[4]{\frac{2}{3}} = \ln \left(\frac{2}{3}\right)^{1/4} = \frac{1}{4} \ln\left(\frac{2}{3}\right)$ | $\log_a M^r = r \log_a M$ (Property vii)

$= \frac{1}{4}(\ln 2 - \ln 3)$ | $\ln(M/N) = \ln M - \ln N$ (Property vi)

$= \frac{1}{4}(a - b)$

46. Write the expression as a sum and/or difference of logarithms. Express powers as factors.

$\log_2\left(\frac{a}{b^2}\right)$ $a>0,\ b>0$

$= \log_2 a - \log_2 b^2$ | $\log_a(M/N) = \log_a M - \log_a N$ (Property vi)

$= \log_2 a - 2\log_2 b$ | $\log_a M^r = r\log_a M$ (Property vii)

53. $\ln\left[\frac{x^2-x-2}{(x+4)^2}\right]^{1/3}$, $x>2$

$= \frac{1}{3}\ln\left[\frac{x^2-x-2}{(x+4)^2}\right]$ | $\log_a M^r = r\log_a M$ (Property vii)

$= \frac{1}{3}\left\{\ln(x^2-x-2) - \ln(x+4)^2\right\}$ | $\log_a(M/N) = \log_a M - \log_a N$ (Property vi)

$= \frac{1}{3}\left[\ln(x-2)(x+1) - \ln(x+4)(x+4)\right]$

$= \frac{1}{3}\left[\ln(x-2) + \ln(x+1) - \ln(x+4) - \ln(x+4)\right]$

| x^2-x-2
$= x^2-2x+x-2$
$= x(x-2)+1(x-2)$
$= (x-2)(x+1)$

64. Write the expression as a single logarithm.

$\log\left(\frac{x^2+2x-3}{x^2-4}\right) - \log\left(\frac{x^2+7x+6}{x+2}\right)$

$= \log\left[\left(\frac{x^2+2x-3}{x^2-4}\right)\Big/\left(\frac{x^2+7x+6}{x+2}\right)\right]$

$= \log\left[\frac{x^2+2x-3}{x^2-4}\cdot\frac{x+2}{x^2+7x+6}\right]$

$= \log\left[\frac{(x-3)\cancel{(x-1)}}{\cancel{(x+2)}(x-2)}\cdot\frac{\cancel{(x+2)}}{(x+6)(x+1)}\right] = \log\left[\frac{(x-3)}{(x+6)(x+1)}\right]$

| x^2+2x-3
$= x^2+3x-x-3$
$= x(x+3)-1(x-3)$
$= (x-3)(x-1)$

x^2+7x+6
$= x^2+6x+x+6$
$= x(x+6)+1(x+6)$
$= (x+6)(x+1)$

61. Write the expression as a single logarithm

$\log_4(x^2-1) - 5\log_4(x+1)$

$= \log_4(x^2-1) - \log_4(x+1)^5$

$= \log_4\left[\dfrac{x^2-1}{(x+1)^5}\right]$

$= \log_4\left[\dfrac{\cancel{(x+1)}(x-1)}{(x+1)^{\cancel{5}\,4}}\right]$

$= \log_4\left[\dfrac{(x-1)}{(x+1)^4}\right]$

88. Express y as a function of x. The constant c is a positive number.

$\ln y = \ln(x+c)$

$\therefore y = x+c$
