6.1 Composite functions

Given two functions $f(x)$ and $g(x)$, the composit function, denoted by $f \circ g$ (read as "f composed with g"), is defined by

$$(f \circ g)(x) = f(g(x))$$

Evaluating composit functions

Problems

13. For the given functions f and g find:

(a) $(f \circ g)(4)$ (b) $(g \circ f)(2)$ (c) $(f \circ f)(1)$ (d) $(g \circ g)(0)$

$$f(x) = 4x^2 - 3, \quad g(x) = 3 - \frac{1}{2}x^2$$

(a) $$(f \circ g)(4) = f(g(4)) = f(-5) = 4 \cdot (-5)^2 - 3 = 100 - 3 = 97$$

$$g(x) = 3 - \frac{1}{2}x^2$$
$$\Rightarrow g(4) = 3 - \frac{1}{2}(4)^2 = 3 - \frac{16}{2} = -5$$

(b) $$(g \circ f)(2) = g(f(2)) = g(13) = 3 - \frac{1}{2}(13)^2 = 3 - \frac{169}{2} = \frac{6 - 169}{2} = \frac{-163}{2}$$

$$f(2) = 4 \cdot (2)^2 - 3 = 16 - 3 = 13$$

(c) $(f \circ f)(1) = f(f(1))$

$= f(1)$

$= 1$

$f(1) = 4 \cdot (1)^2 - 3$

$= 4 - 3$

$= 1$

(d) $(g \circ g)(0) = g(g(0))$

$= g(3)$

$= 3 - \frac{1}{2}(3)^2$

$= 3 - \frac{9}{2} = \frac{6-9}{2} = -\frac{3}{2}$

$g(x) = 3 - \frac{1}{2}x^2$

$g(0) = 3 - \frac{1}{2}(0)^2$

$= 3$

15. $f(x) = \sqrt{x}$; $g(x) = 2x$

(a) $(f \circ g)(4) = f(g(4))$

$= f(8)$

$= \sqrt{8} = \sqrt{\underline{2 \times 2} \times 2}$

$= 2\sqrt{2}$

$g(x) = 2x$

$\Rightarrow g(4) = 2 \cdot 4$

$= 8$

(b) $(g \circ f)(2) = g(f(2))$

$= g(\sqrt{2})$

$= 2\sqrt{2}$

$f(x) = \sqrt{x}$

$\Rightarrow f(2) = \sqrt{2}$

(c) $(f \circ f)(1) = f(f(1))$

$= f(1)$

$= 1$

$f(x) = \sqrt{x}$

$\Rightarrow f(1) = \sqrt{1} = 1$

(d) $(g \circ g)(0) = g(g(0))$

$= g(0)$

$= 0$

$g(x) = 2x$

$\Rightarrow g(0) = 0$

17. $f(x) = |x|$; $g(x) = \frac{1}{x^2+1}$

(a) $(f \circ g)(4) = f(g(4))$

$= f\left(\frac{1}{17}\right)$

$= \left|\frac{1}{17}\right|$

$= \frac{1}{17}$, As absolute value of a positive number is always positive.

$g(x) = \frac{1}{x^2+1}$

$\Rightarrow g(4) = \frac{1}{4^2+1}$

$= \frac{1}{16+1} = \frac{1}{17}$

(b) $(g \circ f)(2)$

$= g(f(2)) = g(2)$

$= \frac{1}{2^2+1} = \frac{1}{5}$

$f(x) = |x|$

$\Rightarrow f(2) = |2|$

$= 2$

c) $(f \circ f)(1)$

$= f(f(1)) = f(1) = 1$

$f(x) = |x|$

$\Rightarrow f(1) = |1| = 1$

d) $(g \circ g)(0)$

$= g(g(0)) = g(1)$

$= \frac{1}{1^2+1} = \frac{1}{2}$

$g(x) = \frac{1}{x^2+1}$

$\Rightarrow g(0) = \frac{1}{0^2+1} = 1$

Finding the domain of composite functions

Problem

25. Find the domain of the composite function $f \circ g$.

$f(x) = \sqrt{x}$; $g(x) = 2x+3$.

The first task is to find the domains of each function separately, which is easy, as you know.

Domain of $f(x)$: All positive real numbers $(\mathbb{R}^+)$, because square root of negative numbers are not defined.

Domain of $g(x)$: All real numbers ($\mathbb{R}$)

Now, you need to find the domain of the composit function $f \circ g = (f \circ g)(x) = f(g(x))$. As, $g(x)$ is allowed over all the real numbers, there is no restriction on $g(x)$.

Now, $f(g(x)) = f(2x+3)$
$= \sqrt{2x+3}$

$f(x) = \sqrt{x}$
$f(2x+3) = \sqrt{2x+3}$

Now, find the domain of the function $\sqrt{2x+3}$.

Naturally, $(2x+3) \geq 0$

$\Rightarrow 2x \geq -3$

$\Rightarrow x \geq -\frac{3}{2}$

$\therefore$ The domain of $f(g(x))$ is ~~all real~~ $x \geq \frac{3}{2}$.

42. For the given functions f and g, find:

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

State the ~~function~~ domain of each composite function.

$f(x) = x^2+4$; $g(x) = \sqrt{x-2}$

- Domain of $f(x)$: All real numbers ($\mathbb{R}$)
- Domain of $g(x)$: $(x-2) \geq 0$
  $\Rightarrow x \geq 2$

(a) $f \circ g = f(g(x)) = f(\sqrt{x-2})$
$= (\sqrt{x-2})^2 + 4$
$= x-2+4 = x+2$

$f(x) = x^2+4$
$\Rightarrow f(\sqrt{x-2}) = (\sqrt{x-2})^2 + 4$

Domain of $f \circ g$:

~~Domain~~ Domain of $g(x)$ is $\{x \mid x \geq 2\}$, which means all real numbers which are greater than or equal to 2. Therefore, the composit function is also not allowed in the range $x \leq 2$.

So, up to this point, we find the domain of $f \circ g$ to be $\{x \mid x \geq 2\}$.

Now look at the function $f \circ g$

$$f \circ g = x + 2.$$

Clearly $(x+2)$ is allowed for any real number x.

$\therefore$ The domain of $f \circ g$ is $\{x \mid x \geq 2\}$.

(b) $g \circ f = g(f(x)) = g(x^2+4)$

$$= \sqrt{x^2+2}$$

$g(x) = \sqrt{x-2}$

$g(x^2+4) = \sqrt{x^2+4-2}$

$= \sqrt{x^2+2}$

Domain of $g \circ f$:

Domain of $f(x)$ is : All real numbers.

Domain of $g \circ f$: $x^2 + 2 \geq 0$

$\Rightarrow x^2 \geq -2$

$\Rightarrow x \geq \pm\sqrt{-2} \Rightarrow$ Complex number

That means the restriction on x is on complex number, however, all real numbers are allowed.

$\therefore$ Domain of $g \circ f$: all real numbers.

(c) $f \circ f = f(f(x)) = f(x^2+4)$ | $f(x) = x^2+4$ $\Rightarrow f(x^2+4) = (x^2+4)^2+4$

$= (x^2+4)^2+4$

$= x^4+8x^2+16+4$

$= x^4+8x^2+20.$

Domain of $f(x)$: All real numbers.

Domain of $f \circ f$: All real numbers.

(d) $g \circ g = g(g(x)) = g(\sqrt{x-2})$ | $g(x) = \sqrt{x-2}$ $g(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$

$= \sqrt{\sqrt{x-2}-2}.$

Domain of $g(x)$: $x \geq 2$.

Domain of $\sqrt{\sqrt{x-2}-2}$: $\sqrt{x-2} \geq 2$

$\Rightarrow x-2 \geq 2^2$

$\Rightarrow x-2 \geq 4$

$\Rightarrow x \geq 6.$

$\therefore$ Domain of $g \circ g = \{x \mid x \geq 2, x \geq 6\}$

$= \{x \mid x \geq 6\}.$

44. For the given functions f and g, find:

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

State the domain of each composite function.

$f(x) = \dfrac{2x-1}{x-2}$; $g(x) = \dfrac{x+4}{2x-5}$

Domain of $f(x)$: All real numbers except $x \neq 2$.

$\{x \mid x \neq 2\}$.

Domain of $g(x)$: $2x-5=0$ $\Rightarrow x = 5/2$ } Means all real numbers except $x = 5/2$.

$\{x \mid x \neq 5/2\}$.

(a) $f \circ g = f(g(x)) = f\left(\frac{x+4}{2x-5}\right)$.

$f(x) = \frac{2x-1}{x-2}$

$\Rightarrow f\left(\frac{x+4}{2x-5}\right) = \frac{2 \cdot \frac{x+4}{2x-5} - 1}{\frac{x+4}{2x-5} - 2}$.

$$= \frac{2 \cdot \frac{x+4}{2x-5} - 1}{\frac{x+4}{2x-5} - 2}$$

$$= \frac{\frac{\cancel{2x} + 8 - \cancel{2x} + 5}{2x-5}}{\frac{x+4-2(2x-5)}{2x-5}}$$

$$= \frac{13/(2x-5)}{\frac{x+4-4x+10}{2x-5}} = \frac{13}{\cancel{(2x-5)}} \times \frac{\cancel{(2x-5)}}{-3x+14}$$

$$= \frac{13}{14-3x}.$$

Domain of $g(x)$: $\{x \mid x \neq 5/2\}$.

Domain of $\frac{13}{14-3x}$: $\left.\begin{array}{l} 14-3x=0 \\ \Rightarrow x = \frac{14}{3} \end{array}\right\}$ $\{x \mid x \neq \frac{14}{3}\}$.

$\therefore$ Domain of $f \circ g$: $\{x \mid x \neq \frac{5}{2}, x \neq \frac{14}{3}\}$, which means domain of x is all real numbers, except $x = \frac{5}{2}$ and $x = \frac{14}{3}$.

(b) $g \circ f = g(f(x)) = g\left(\frac{2x-1}{x-2}\right)$

$g(x) = \frac{x+4}{2x-5}$

$g\left(\frac{2x-1}{x-2}\right) = \frac{\frac{2x-1}{x-2} + 4}{2 \cdot \frac{2x-1}{x-2} - 5}$

$$= \frac{\frac{2x-1}{x-2} + 4}{\frac{4x-2}{x-2} - 5}$$

$$g \circ f = \frac{\frac{2x-1+4(x-2)}{x-2}}{\frac{4x-2-5(x-2)}{x-2}} = \frac{\frac{2x-1+4x-8}{x-2}}{\frac{4x-2-5x+10}{x-2}}$$

$$= \frac{(6x-9)}{\cancel{(x-2)}} \cdot \frac{\cancel{(x-2)}}{(8-x)}$$

$$= \frac{6x-9}{8-x}.$$

Domain of $f(x)$: $\{x \mid x \neq 2\}$

Domain of $\frac{6x-9}{8-x}$: $\{x \mid x \neq 8\}$.

$\therefore$ Domain of $g \circ f$: $\{x \mid x \neq 2, x \neq 8\}$.

(c) $f \circ f = f(f(x)) = f\left(\frac{2x-1}{x-2}\right)$

$$f(x) = \frac{2x-1}{x-2}$$

$$f\left(\frac{2x-1}{x-2}\right) = \frac{2 \cdot \frac{2x-1}{x-2} - 1}{\frac{2x-1}{x-2} - 2}.$$

$$= \frac{\frac{4x-\cancel{2}-x+\cancel{2}}{x-2}}{\frac{2x-1-2(x-2)}{x-2}}$$

$$= \frac{\cancel{3}x}{\cancel{(x-2)}} \cdot \frac{\cancel{(x-2)}}{\cancel{3}} = x$$

Domain of $f(x)$: $\{x \mid x \neq 2\}$

Domain of x : all real numbers.

$\therefore$ Domain of $f \circ f$: $\{x \mid x \neq 2\}$.

(d) $g \circ g = g(g(x)) = g\left(\frac{x+4}{2x-5}\right)$

$g(x) = \frac{x+4}{2x-5}$

$\Rightarrow g\left(\frac{x+4}{2x-5}\right) = \dfrac{\frac{x+4}{2x-5}+4}{2\cdot\frac{x+4}{2x-5}-5}$

$$= \dfrac{\frac{x+4+4(2x-5)}{2x-5}}{\frac{2x+8-5(2x-5)}{2x-5}}$$

$$= \frac{x+4+8x-20}{\cancel{(2x-5)}} \cdot \frac{\cancel{(2x-5)}}{2x+8-10x+25}$$

$$= \frac{9x-16}{33-8x}$$

Domain of $g(x)$: $\{x \mid x \neq \frac{5}{2}\}$

Domain of $\frac{9x-16}{33-8x}$: $\{x \mid x \neq \frac{33}{8}\}$

$\therefore$ Domain of $g \circ g$: $\{x \mid x \neq \frac{5}{2}, x \neq \frac{33}{8}\}$.

6.2 One-to-one and inverse functions :

A function is one-to-one if any two different inputs in the domain correspond to two different outputs in the range. That is, if x_1 and x_2 are two different inputs of a function f, then f is one-to one if $f(x_1) \neq f(x_2)$.

Horizontal line test

If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

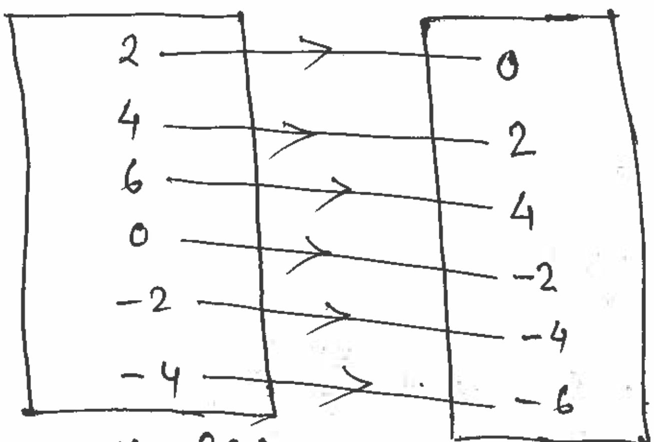
For example $f(x) = x^2$ is not a one-to-one function, whereas, $f(x) = x^3$ is a one-to-one function.

Inverse function

$y = f(x) = x + 2$ is a one-to-one function, because there is one value of y corresponding to each value of x. Let us make a table –

y	2	4	6	0	−2	−4
x	0	2	4	−2	−4	−6

We make a diagram out of it.



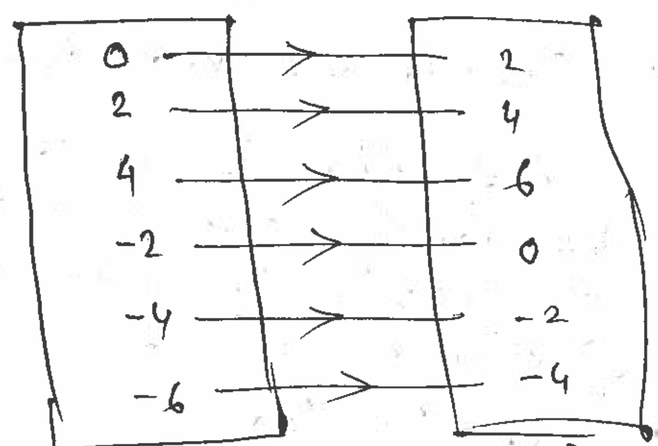
$y = f(x)$ $\qquad$ x

or, in the set notation, we can write the one-to-one function in the following way –

$\{(2,0), (4,2), (6,4), (0,-2), (-2,-4), (-4,-6)\}$.

Now we interchange the position as shown in the figure. Now, the inverse function is written in the following way –

$\{(0,2), (2,4), (4,6), (-2,0), (-4,-2), (-6,-4)\}$.



x $\qquad$ $y = f(x)$

So, the one-to-one function is written in the ordered pair of (x,y), while the inverse function, denoted as f^{-1}, is written as the inversed ~~order~~ ordered pair (y,x).

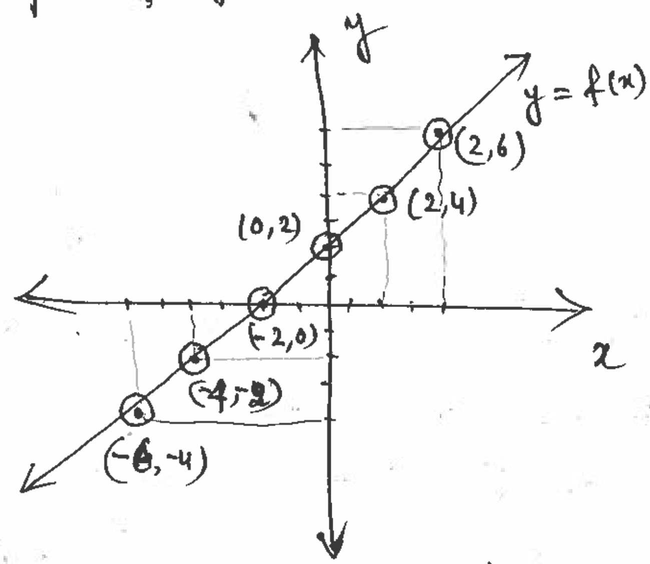
$\therefore \quad y = f(x)$

$\Rightarrow \quad x = f^{-1}(y)$

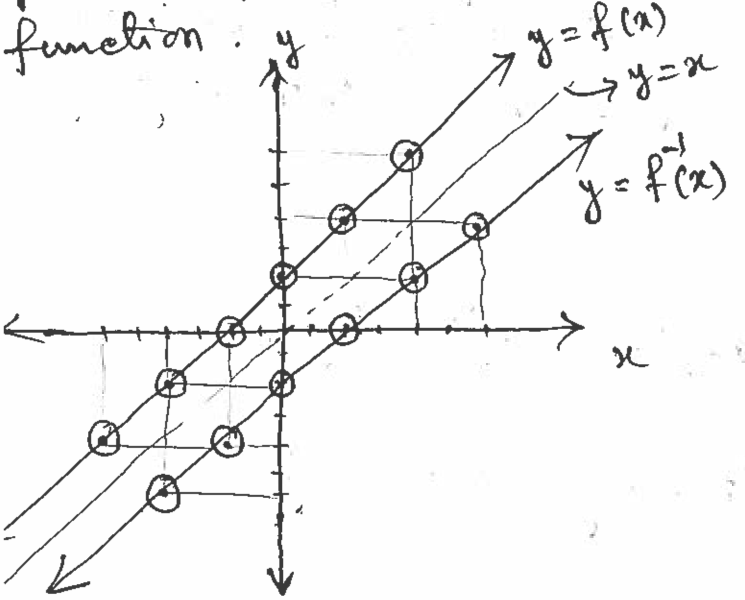
The domain of the function $y = f(x)$ is, therefore, $\{2, 4, 6, 0, -2, -4\}$ and range $= \{0, 2, 4, -2, -4, -6\}$. While, the domain and range of the inverse function are $\{0, 2, 4, -2, -4, -6\}$ and $\{2, 4, 6, 0, -2, -4\}$ respectively.

Graph of the inverse function from the graph of the function

Let us first plot the graph of the function. We considered $y = f(x) = x + 2$.



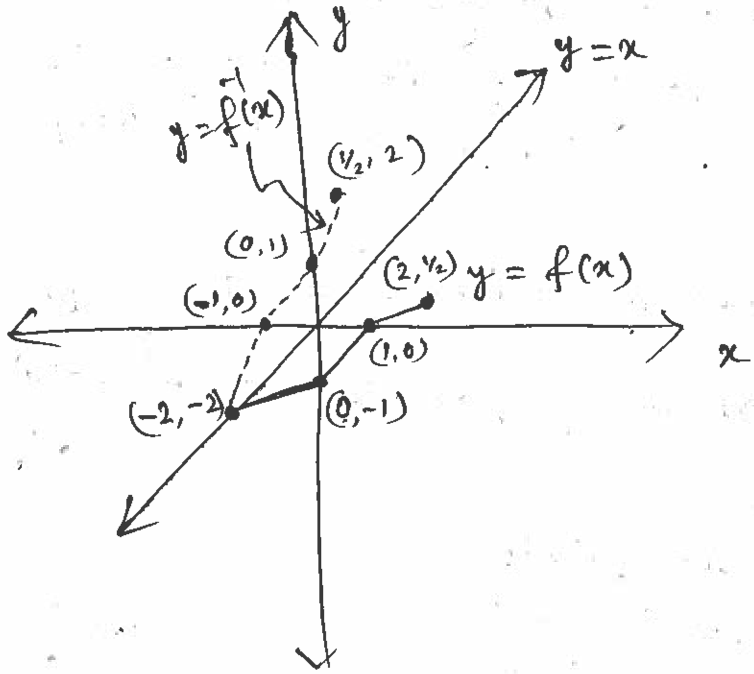
While, you know the points of the function, the inverse of the function is formed by of the function is interchanging the points. Then one can plot the interchanged points to plot the inverse of the function.



The graph of $f^{-1}(x)$ is obtained by taking a mirror reflection of the graph of $f(x)$ with respect to the $y = x$ line.

Problems :

44. The graph of one-to-one function f is given. Draw the graph of the inverse function f^{-1}. For convenience, the graph of $y = x$ is also given.



56. The function $f(x) = x^2 + 9$, $x \geq 0$ is one-to-one. Find its inverse and check your answer. Graph f, f^{-1} and $y = x$ on the same coordinate axes.

The inverse of the function $f(x) = x^2 + 9$ is obtained by first replacing '$f(x)$' by 'y' and then interchanging x and y.

$$f(x) = x^2 + 9$$

$$y = x^2 + 9 \quad \text{(Replace } f(x) \text{ by } y\text{)}.$$

$$\boxed{x = y^2 + 9} \quad \text{(Interchange } x \text{ and } y\text{)}.$$

$\Downarrow$

Inverse function.

To find the explicit form of the inverse function, one needs to solve the equation for y.

$$x = y^2 + 9$$
$$\Rightarrow y^2 = x - 9$$
$$\Rightarrow y = \sqrt{x-9}.$$

$\therefore$ The inverse is $f^{-1}(x) = \sqrt{x-9}$; $x \geq 9$.

<u>check</u> :

$f^{-1}(f(x)) = f^{-1}(x^2+9)$

$= x$, where x in in the domain of f

$f^{-1}(x) = \sqrt{x-9}$

$f^{-1}(x^2+9) = \sqrt{x^2+\cancel{9}-\cancel{9}}$

$= x$

$f(f^{-1}(x)) = f(\sqrt{x-9})$.

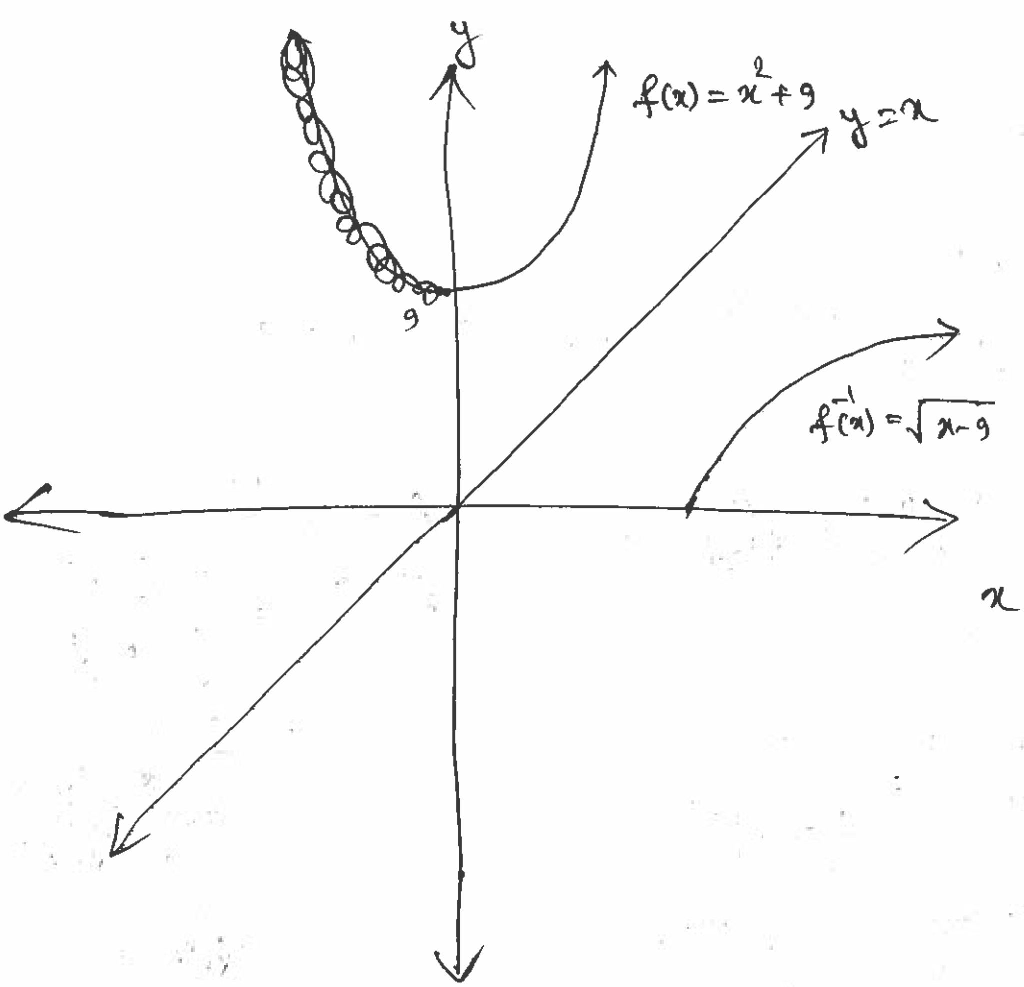
$= x$, where x is in the domain of f^{-1}.

$f(x) = x^2+9$

$f(\sqrt{x-9}) = (\sqrt{x-9})^2 + 9$

$= x - 9 + 9$

$= x$



The left hand side of the graph of $f(x) = x^2+9$ is not plotted in the graph. The reason is because, the domain of the function f is $x \geq 0$, as given in the problem.

62. The function $f(x) = \frac{4}{2-x}$ is one-to-one. Find its inverse and check your answer.

$$f(x) = \frac{4}{2-x}, \quad \text{Domain is } x \neq 2.$$

Replace $f(x)$ by y

$$y = \frac{4}{2-x}$$

Interchange x and y

$$x = \frac{4}{2-y} \Rightarrow \underline{\text{Inverse function}}$$

Solve for y, to obtain the explicit form of the inverse function.

$$x = \frac{4}{2-y}$$

$$\Rightarrow (2-y) = \frac{4}{x}$$

$$\Rightarrow -y = \frac{4}{x} - 2$$

$$\Rightarrow y = 2 - \frac{4}{x} = \frac{2x-4}{x}. \quad \text{Domain is } x \neq 0.$$

<u>check</u>

$$f^{-1}(f(x)) = f^{-1}\left(\frac{4}{2-x}\right) = x,$$

where $x \neq 2$.

$$f(f^{-1}(x)) = f\left(\frac{2x-4}{x}\right) = \frac{4}{2 - \frac{2x-4}{x}}$$

$$= \frac{4}{\frac{\cancel{2x}-\cancel{2x}+4}{x}} = \frac{4}{\frac{4}{x}} = \frac{\cancel{4}}{1} \times \frac{x}{\cancel{4}} = x$$

where $x \neq 0$.

$$f^{-1}(x) = \frac{2x-4}{x}$$

$$f^{-1}\left(\frac{4}{2-x}\right) = \frac{2 \cdot \frac{4}{2-x} - 4}{4/(2-x)}$$

$$= \frac{\frac{\cancel{8} - \cancel{8} + 4x}{2-x}}{4/(2-x)}$$

$$= \frac{4x}{\cancel{(2-x)}} \cdot \frac{\cancel{(2-x)}}{\cancel{4}}$$

$$= x$$