5.1 Polynomial function

We have already learnt about the linear function and ~~a~~ quadratic function in the last lectures. However, they are the examples of polynomial functions as some special cases. For instance, linear functions are polynomial functions of degree 1, whereas quadratic functions are polynomial functions of degree 2. We do not know anything about the degree of a polynomial. However, before that, let us see the explicit form of a polynomial function, which is shown below.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 ,$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are real numbers and n is a non-negative integer. The above function is a general form of a polynomial function, however, in the absence of one or more than one terms in the series, the function can still be said a polynomial function. Certainly, a polynomial function must be a function of a single variable.

The <u>degree of a polynomial</u> is the largest power of n that appears in the function. Now you see why a linear function is a polynomial function of degree 1, whereas a quadratic function is a polynomial function of degree 2.

Examples:

1. $f(x) = 5x^5 + 2x + 1$: polynomial function of degree 5.
2. $f(x) = \dfrac{x+5}{x^2-1}$: not a polynomial function.
3. $f(x) = 0$: zero polynomial function, not assigned to a degree.

4. $f(x) = \sqrt{x}$: not a polynomial function, because the power of the variable x is $1/2$, which is not a positive integer.

5. $f(x) = 5$: a polynomial function of degree 0, since $f(x) = 5 = 5 \cdot x^0$.

Problems

18. Determine which functions are polynomial functions. For those that are, state the degree. For those that are not, tell why not:

$f(x) = 3 - \frac{1}{2}x$: is clearly a polynomial function of degree 1, since the maximum power of the variable x is 1.

21. $g(x) = x^{3/2} - x^2 + 2$: is not a polynomial fun. The reason is because the one of the powers of the variable is not an integer.

25. $G(x) = 2(x-1)^2(x^2+1)$

$= 2(x^2 - 2x + 1)(x^2+1)$

$= (2x^2 - 4x + 2)(x^2+1)$

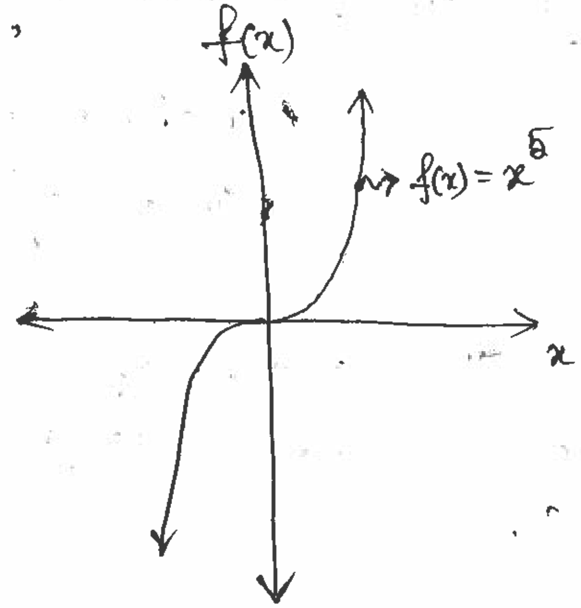
$= 2x^4 + 2x^2 - 4x^3 - 4x + 2x^2 + 2$

$= 2x^4 - 4x^3 + 4x^2 - 4x + 2$

is a nice example of a polynomial of degree 4.

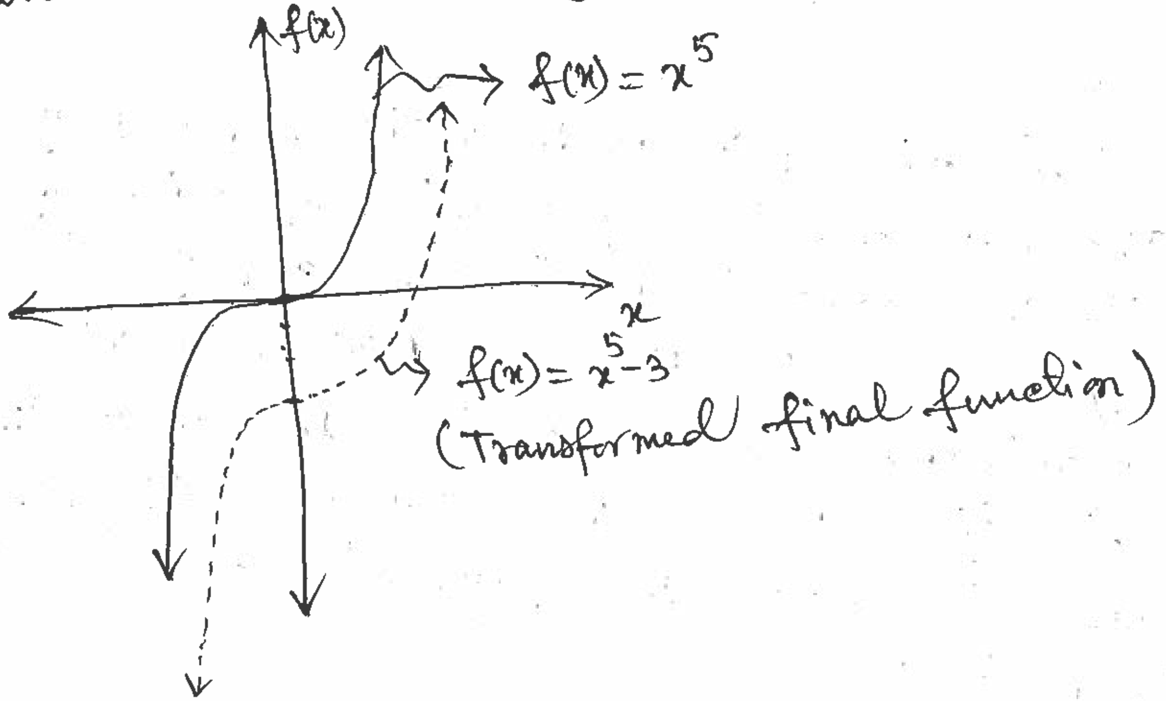
29. Use transformations of the graph of $y = x^5$ to graph the function $f(x) = x^5 - 3$.

Suppose the graph of the function $f(x) = x^5$ is known, as depicted in the following figure, then it is quite easy to transform the function using the rules that were discussed in one of our previous lectures.



In this particular case, the function will be shifted down along the y-axis by 3-units.

$\therefore f(x) = x^5 - 3$ is a new function as plotted in the following figure.



5.2 Properties of Rational Functions:

If you are well aware of the definition of a rational number as discussed in lecture-1, you ~~are~~ know what a Rational Function is. Ratios of integers are called rational numbers. Quite similarly, ratios of polynomial functions are called rational functions.

Domain of rational functions:

A rational function, is clearly of a form

$$R(x) = \frac{p(x)}{q(x)}, \text{ where } p(x) \text{ and } q(x) \text{ are polynomial functions.}$$

The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Problems

13. Find the domain of each rational function.

$$R(x) = \frac{4x}{x-3}$$

The rational function R(x) is the ratio of two polynomial functions. The numerator is a polynomial function of degree 1, while the denominator is also a polynomial function of degree 1. However, R(x) is undefined when the denominator is zero. Therefore, the domain of the rational function is all real numbers except $x = 3$, because for $x = 3$, the polynomial function in the denominator is zero.

15. $$H(x) = \frac{-4x^2}{(x-2)(x+4)}$$

We have to evaluate, the value/values of x for which the polynomial function in the denominator is zero.

$(x-2)(x+4)=0$

The solutions are

either, $x-2=0$ or, $x+4=0$

$\Rightarrow x=2$ $\Rightarrow x=-4$.

$\therefore$ The function $(x-2)(x+4)$ is equals zero for $x=2$ and $x=-4$.

The domain of $H(x)$ is therefore <u>all real numbers except $x=2$ and $x=-4$.</u>

<u>21</u>. $H(x) = \dfrac{3x^2+x}{x^2+4}$

Solve, $x^2+4=0$

$\Rightarrow x^2 = -4$

$\Rightarrow x = \pm\sqrt{-4}$

$\therefore x = +2i$ and $-2i$, where $i=\sqrt{-1}$ is a complex number.

$\therefore H(x)$ is defined for all real numbers of x.

That is why the domain is all real numbers.

Asymptotes

Let us start our discussion with a very simple example of a function $f(x) = \frac{1}{x^2}$.

You know how to plot the function. You can either make a table x vs $f(x)$ and plot the points on the coordinate plane. Otherwise, you can follow the standard procedure that we have discussed in last few lectures.

One thing is clear that $f(x)$ is undefined when $x=0$. On the other hand $f(x)$ approaches to zero when $x=\infty$.

To find y-intercepts, put $x=0$, which makes the function $f(x)$ undefined. Therefore, there is no y-intercept.

While put $y=0$, $f(x)=0$ has no solution either. Therefor, the function has no x-intercept as well.

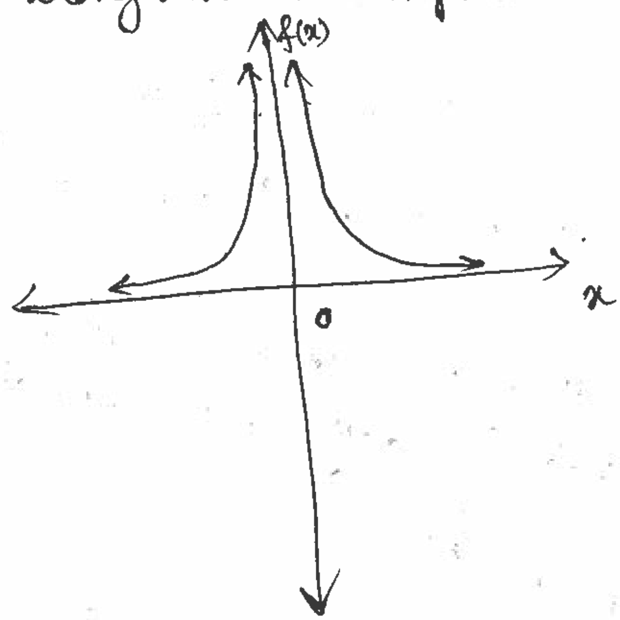
let us now check the symmetry of the function.

$$f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = f(x).$$

$f(x)$ is therefore an even function and so its graph is symmetric with respect to the y-axis.

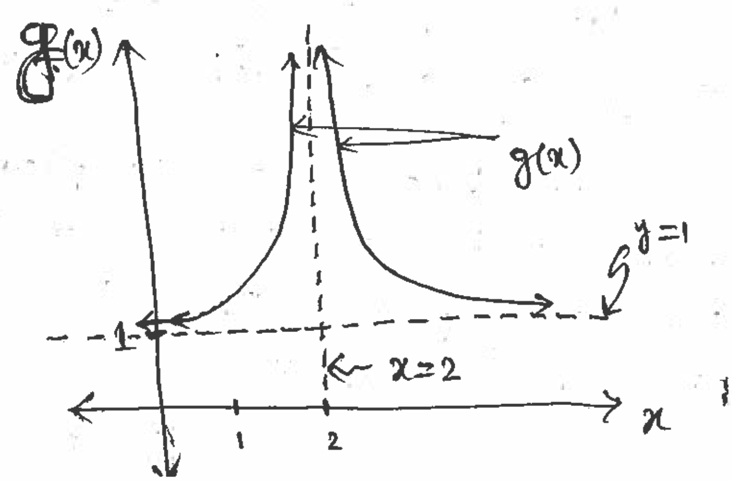
We now plot the function along with two important properties —

$$\left.\begin{aligned} &\lim_{x\to 0} f(x) = \infty \\ \text{and } &\lim_{x\to\infty} f(x) = 0 \end{aligned}\right\}$$



Now, we can apply the rules of the transformations to plot the function $g(x) = \frac{1}{(x-2)^2} + 1$.

The resulting graph of $g(x)$ will be shifted 1 unit along the +ve y-axis, and 2 units along +ve x-axis.



The roles of the lines $x=2$ and $y=1$ are important here. As $g(x)$ approaches to infinite as x approaches to 2, <u>$x=2$ is called the vertical asymptote of the function $g(x)$.</u>

On the other hand $g(x)$ is approaches to 1, when x approaches to ∞. The line <u>$y=1$ is, therefore, called the horizontal asymptote of the function $g(x)$</u>.

Similar analysis says that $x=0$ is the vertical asymptote of the function $f(x)$, whereas $y=0$ is the horizontal asymptote of the function $f(x)$.

~~Note that for the cases of vertical and horizontal asymptotes, the function never intersect the asymptotes. However, there is a third possibility, where~~

Note that the vertical asymptote is parallel to the y-axis, whereas the horizontal asymptote is parallel to x-axis. There is the third possibility, where the asymptote is not parallel to x or y axis, is known as slant or <u>oblique asymptote</u>. The graph of a function never intersects with the vertical asymptote, while the graph of a function may intersect with vertical and oblique asymptotes.

<u>Find the ~~vertical~~ asymptotes of a rational function</u>

Finding a <u>vertical asymptote</u> of a rational function $R(x) = \frac{p(x)}{q(x)}$ is quite easy. One needs to just solve the equation <u>$q(x) = 0$</u>. The solution is the vertical asymptote.

While finding the horizontal and oblique asymptote are slighly difficult.

At the first point, one needs to check whether the degree of the polynomial in the numerator is less than that of the degree of the denominator. If that is true then $y=0$ is the horizontal polynomial.

If the statement is not true, that means the degree of the numerator is greater than the denominator, one needs to divide the numerator with the denominator. Let us say

$R(x) = \frac{p(x)}{q(x)}$ is a polynomial function, where degree of $p(x) >$ degree of $q(x)$.

Then divide $p(x)$ by $q(x)$, let us say the quotient is a polynomial $f(x)$ and the remainder is another polynomial $r(x)$. Then one can write $R(x)$ in the form as follows

$$R(x) = f(x) + \frac{r(x)}{q(x)} = \frac{p(x)}{q(x)}.$$

Now you look at the function $f(x)$ (quotient) carefully.

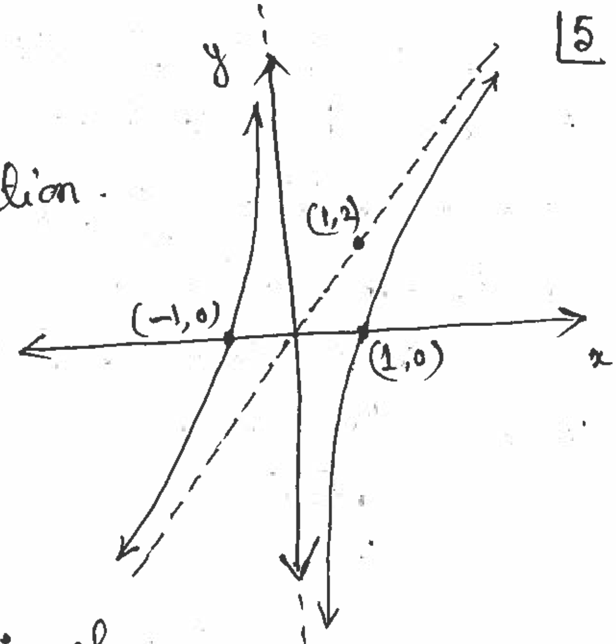
- If $f(x) = b$, a constant, the line $y=b$ is a horizontal asymptote of the graph.

- If $f(x) = ax+b$, with $a \neq 0$ and b = constant, the line $y = ax+b$ is an oblique asymptote of the graph.

- In all other cases, there are no horizontal or oblique asymptotes

Problems

27. Use the graph to find
(a) Domain and range of the function.
(b) Intercepts, if any.
(c) Horizontal asymptotes, if any.
(d) Vertical asymptotes, if any.
(e) Oblique asymptotes, if any.



(a) The function is defined for all values of x except $x=0$. So, the domain of the function is all real numbers except $x=0$

The value of the function $f(x)$ ranges from $(-\infty)$ to $(+\infty)$.
$\therefore$ The range of the function is all real numbers.

(b) Only two x-intercepts at $(-1,0)$ and $(1,0)$. No y-intercept.
(c) No horizontal asymptote.
(d) Vertical asymptote is $x=0$
(e). Only one oblique asymptote. To find the equation of the oblique asymptote, we need to go back to our previous lectures to find the slope and the intercept of the line.
y-intercept is certainly at zero.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{2}{1} = 2$$

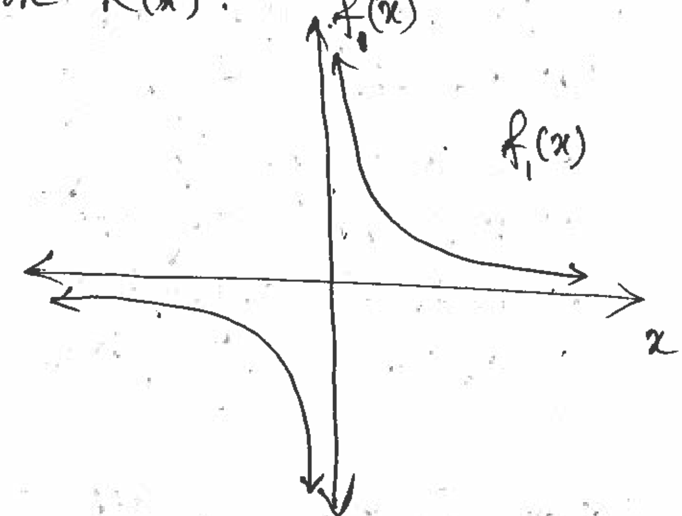
$\therefore$ The equation of the oblique intercept is $y = 2x$.

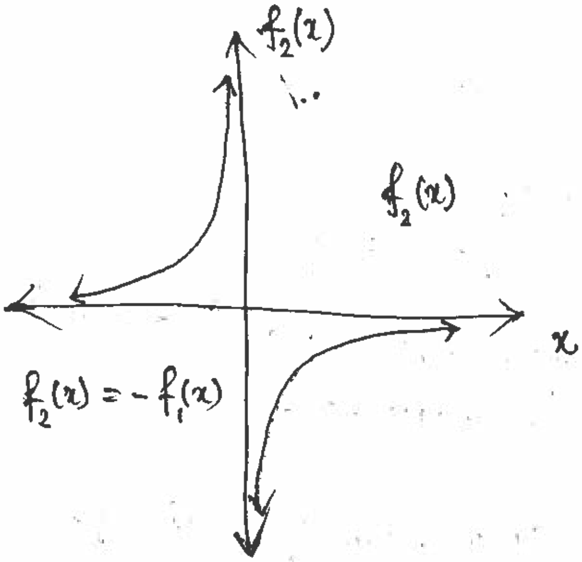
42. Graph the function $R(x) = \dfrac{x-4}{x}$ using transformation.

$$R(x) = \frac{x-4}{x} = 1 - \frac{4}{x}$$

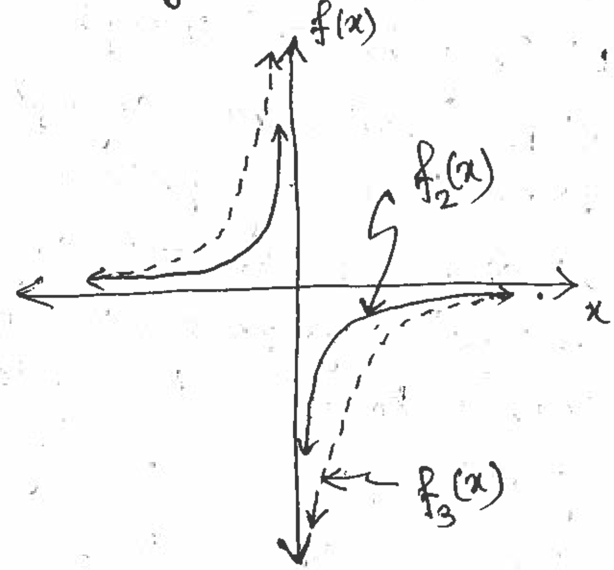
We use the basic function $f_1(x) = \frac{1}{x}$ and then take the transformations to reach the function $R(x)$.

Consider $f_2(x) = \frac{-1}{x}$. According to the transformation rule $f_2(x)$ is just a reflection of $f_1(x)$ with respect to x-axis.

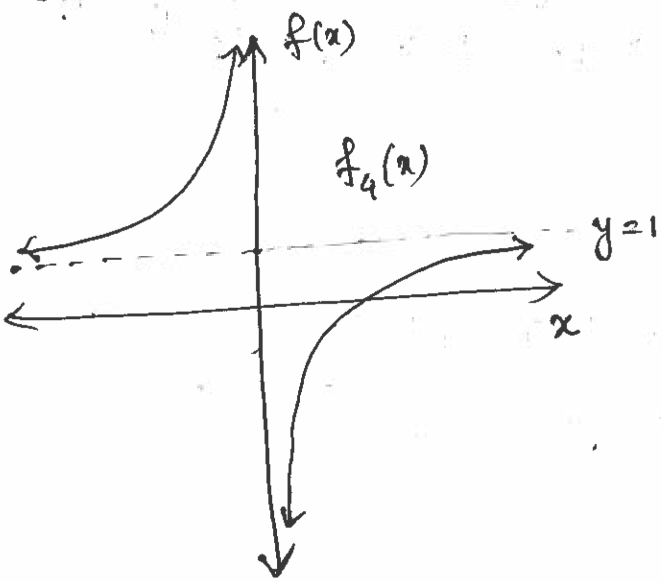




Now consider $f_3(x) = \frac{-4}{x} = 4\,f_2(x)$.
According to the rule, as the function is multiplied by a number, which is greater than 1, the graph will be streached vertically. Each $f_2(x)$ will be multiplied by 4.



$f_4(x) = 1 + f_3(x) = 1 - \frac{4}{x}$.
So, $f_4(x)$ will be shifted 1 unit along the +ve y-axis.



44. Find the vertical, horizontal and oblique asymptoties, if any. $R(x) = \dfrac{3x+5}{x-6}$

Finding vertical asymptotic is easy. One just needs to solve

$x-6=0$

$\Rightarrow x=6 \Leftarrow$ Vertical asymptotic.

To find horizontal and oblique asymptoties, we need to check whether the degree of the numerator is less than the denominator. This is not the case here. So, we have to divide the numerator with the denominator.

$$\begin{array}{r|l} & 3 \\ \hline x-6 & 3x+5 \\ & \underset{(-)}{3x}\ \underset{(+)}{-18} \\ \hline & 23 \end{array}$$

Quetient : 3

Remainder : 23

Then express $R(x)$ in the following form:

$$R(x) = \frac{3x+5}{x-6} = \text{Quetient} + \frac{\text{Remainder}}{x-6}$$

$$= 3 + \frac{23}{x-6}$$

As quetient is a constant, the line $y=3$ is a horizontal asymptotic.

49. $Q(x) = \dfrac{2x^2-5x-12}{3x^2-11x-4}$

Vertical asymptotic: $3x^2-11x-4=0$

$$\therefore x = \frac{11 \pm \sqrt{(11)^2+4\cdot 3\cdot 4}}{2\times 3} = \frac{11\pm 13}{6}$$

$\therefore$ vertical asymptoties are: $x=4$ and $x=\frac{1}{3}$.

~~As the degree of the numerator < degree of denominator, the horizontal asymptotic is $y=0$.~~

~~let us see how:~~

$$\cancel{R(x) = \frac{2x^2-5x-12}{3x^2}}$$

---

degree of numerator = degree of denominator.

$$\begin{array}{r l} & \frac{2}{3} \\ 3x^2-11x-4 \, \big) & 2x^2-5x-12 \\ & 2x^2 - \frac{22}{3}x - \frac{8}{3} \\ & (-) \quad (+) \quad (+) \\ \hline & \frac{7}{3}x - \frac{28}{3} \end{array}$$

Quotient : $\frac{2}{3}$ = Constant

Remainder : $\frac{7}{3}x - \frac{28}{3}$

$$\therefore R(x) = \frac{2x^2-5x-12}{3x^2-11x-4} = \frac{2}{3} + \frac{\frac{7}{3}x - \frac{28}{3}}{3x^2-11x-4}$$

As the quotient is a constant, the horizontal asymptotic is

$$y = \frac{2}{3}.$$

48. $P(x) = \dfrac{4x^2}{x^3-1}$

vertical asymptotic : $x^3-1=0$

$$\Rightarrow x^3 = 1$$

$$\Rightarrow x = \sqrt[3]{1} = 1$$

As the degree of numerator < degree of denominator, the horizontal asymptotic is $y=0$. let us see how –

$P(x) = \dfrac{4x^2}{x^3-1}$. For large value of x, we can consider only the leading term for our calculation

$\therefore P(x) = \dfrac{4x^2}{x^3} = \dfrac{4}{x} \Leftarrow$ horizontal asymptotic is at $y = 0$.

52. $R(x) = \dfrac{8x^2 + 26x - 7}{4x - 1}$

Vertical asymptotic : $4x - 1 = 0$
$\Rightarrow x = \frac{1}{4}$.

$$
\begin{array}{r|l}
 & 2x + 7 \\
4x-1 & 8x^2 + 26x - 7 \\
 & \underset{(-)}{8x^2} \underset{(+)}{-} 2x \\ \hline
 & 28x - 7 \\
 & \underset{(-)}{28x} \underset{(+)}{-} 7 \\ \hline
\end{array}
$$

Quotient : $2x+7$

Remainder : 0

$\therefore R(x) = \dfrac{8x^2 + 26x - 7}{4x - 1} = 2x + 7 + \dfrac{0}{4x-1}$

$= 2x + 7$.

As the quotient is $(2x+7)$, which is a linear function of x, the slant / oblique asymptotic is $y = 2x + 7$.

5·3 Graph of rational function

The principal motivation of this chapter is to analyse the rational function and then plot the function. We will look at the problems to see how -

Problems

8. $R(x) = \dfrac{x}{(x-1)(x+2)}$

Step-1 : Factor the numerator and denominator, if they are not factored. In this case they are factored. Find the domain of $R(x)$. The domain of $R(x)$ in the present problem is all real numbers except $x = 1$ and $x = -2$.

Step-2

Simplify $R(x)$ after doing the factorization. In our case the $R(x)$ is in its lowest term.

Step-3

i) Locate the intercepts of the graph. Since 0 is in the domain of $R(x)$, the y-intercept can be found by putting $x=0$ in $R(x)$. The y-intercept is therefore

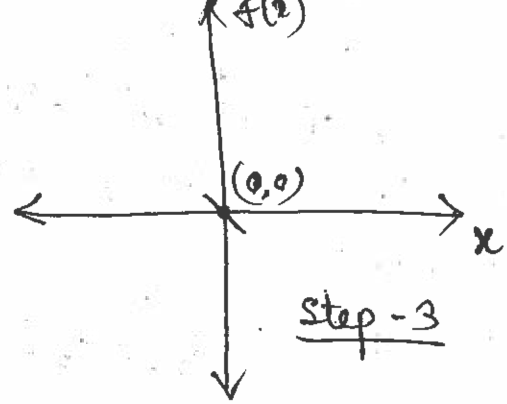
$$R(0) = \frac{0}{(0-1)(0+2)} = \frac{0}{-2} = 0$$

The x-intercepts are found by determining the real zeros of the numerator of R that are in the domain of R. The only zero of the numerator is 0, so the only x-intercept is 0.

(ii) Analyse the behaviour of the graph of R near the x intercepts.

$$\text{Near } x=0 : \quad R(x) = \frac{x}{(0-1)(0+2)} = -\frac{x}{2}$$

clearly the slope of the line is negative. Therefore plot the point ~~point~~ (0,0) and draw a line with negative slope there.

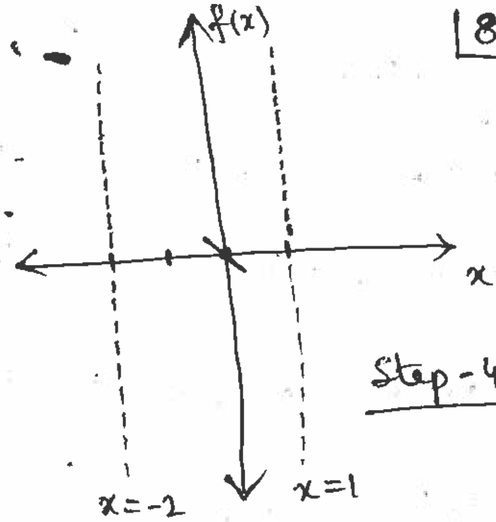


Step-3

Step-4

Locate the vertical asymptotes and graph each of them by a dashed line.

As we know, vertical asymptotes are found by determining the real zeros of the denominator with the rational function in lowest term. In our case $R(x)$ is in the lowest term. So, the vertical asymptotes are at $x=1$ and $x=-2$.



Step-4

Step-5 Locate the horizontal ~~and~~ or oblique asymptote, if exists.

Since, the degree of the numerator is less than the degree of the denominator, the only horizontal asymptote of the graph of $R(x)$ is $y=0$.

Next determine points, if any, at which the graph of $R(x)$ intersect the asymptotes.

To determine if the graph of R intersects the horizontal asymptote, solve the equation $R(x)=0$

$$\therefore \frac{x}{(x-1)(x+2)} = 0$$

$$\Rightarrow x = 0$$

$\therefore$ The graph of $R(x)$ intersects the horizontal asymptote at $x=0$.

Step-6

Use the zeros of the numerator and denominator of $R(x)$ to divide the x-axis into intervals. Determine where the graph of $R(x)$ is above or below the x-axis by choosing a number in each interval and evaluate $R(x)$ at that point. Plot the points that you have ~~just~~ found just now.

Zero of the numerator : 0

Zeros of the denominator : 1, -2

Divide the x-axis into 4-intervals : $(-\infty,-2)$, $(-2,0)$, $(0,1)$, $(1,\infty)$

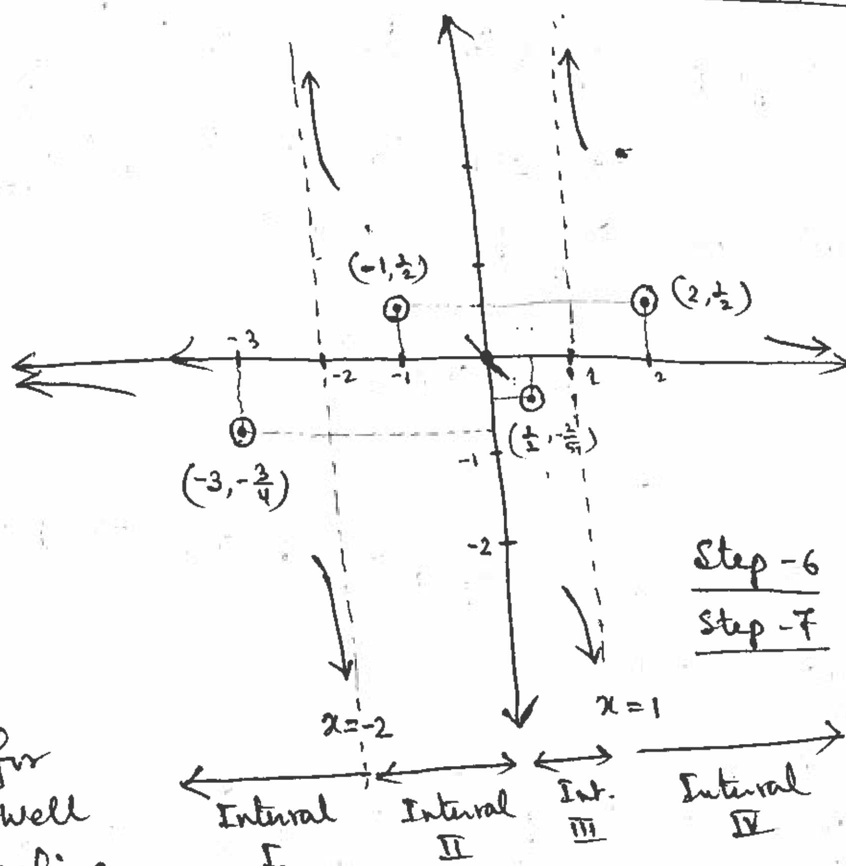
Now construct the following table —

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,1)$	$(1,\infty)$
Number chosen (x)	-3	-1	$\frac{1}{2}$	2
Value of $R(x)$	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{2}{5}$	$\frac{1}{2}$
Location of graph	Below x-axis	above x-axis	below x-axis	above x-axis
Point on graph	$(-3,-\frac{3}{4})$	$(-1,\frac{1}{2})$	$(\frac{1}{2},-\frac{2}{5})$	$(2,\frac{1}{2})$

Step-7

- $y=0$ is the horizontal asymptote and the graph lies below the x-axis for $x<-2$. We can therefore sketch a portion of the graph by placing a small arrow to the far left and under the x-axis.

- Since the line $x=-2$ is a vertical asymptote and the graph lies below the x-axis for $x<-2$, we place an arrow well below the x-axis and approaching the line $x=-2$ from the left.

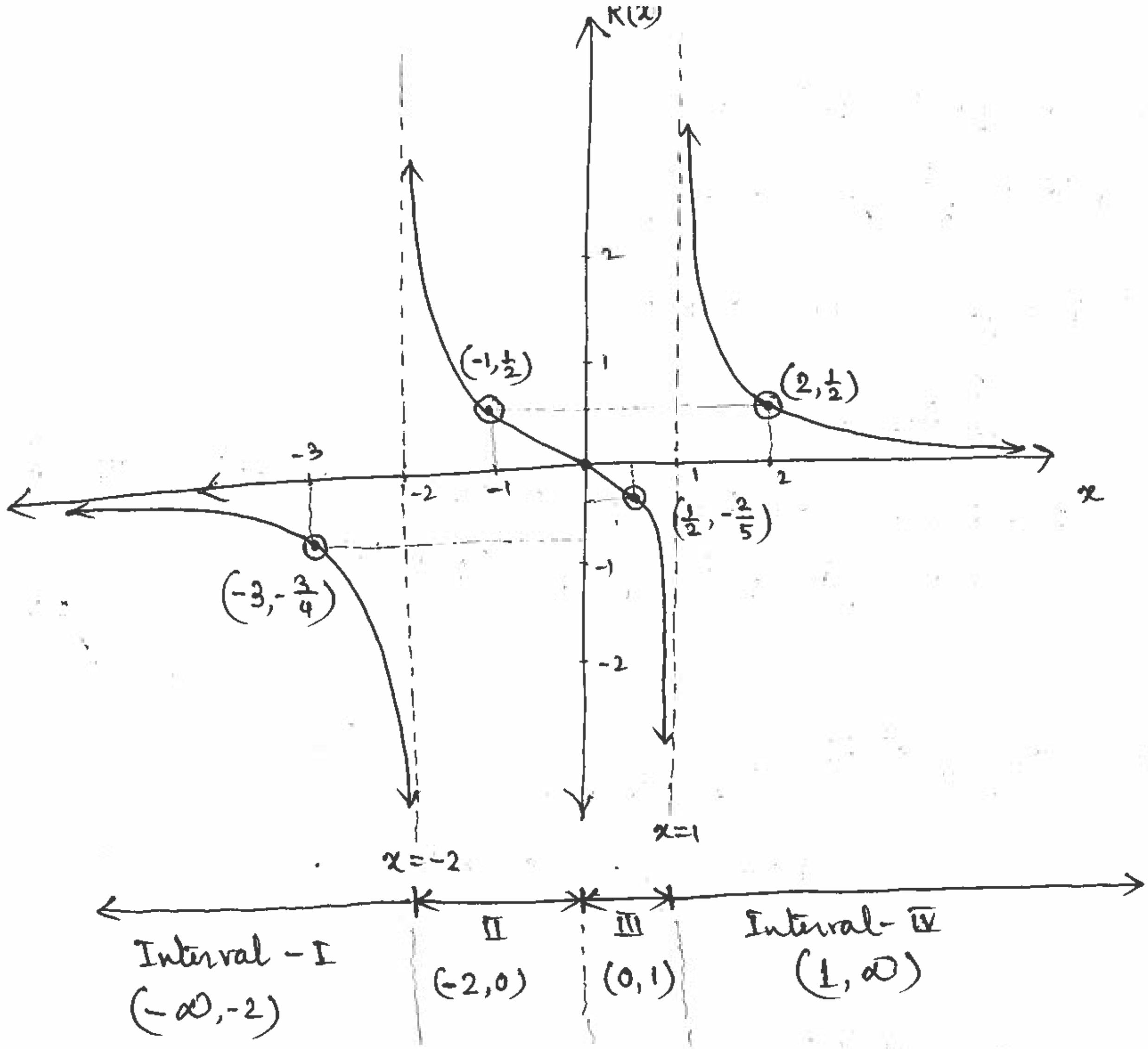


Step-6
Step-7

Similar explanations applied for the other arrows.

Step-8

Plot the graph with all the above explanations. This is of course the final step. The below figure shows the graph of the function $R(x)$.



11. $R(x) = \dfrac{3}{x^2-4}$

Step-1 Factor the denominator :

$$(x^2-4) = (x)^2 - (2)^2 = (x+2)(x-2)$$

Domain of $R(x)$: All real numbers except 2, -2

Step-2
$R(x)$ is already in its simplest form : $R(x) = \dfrac{3}{(x+2)(x-2)}$

Step-3

y intercept : (Put $x=0$)

$\therefore R(0) = \dfrac{3}{(0+2)(0-2)} = -\dfrac{3}{4}$

x intercept : (Put $R(x)=0$)

$$\frac{3}{(x+2)(x-2)} = 0$$

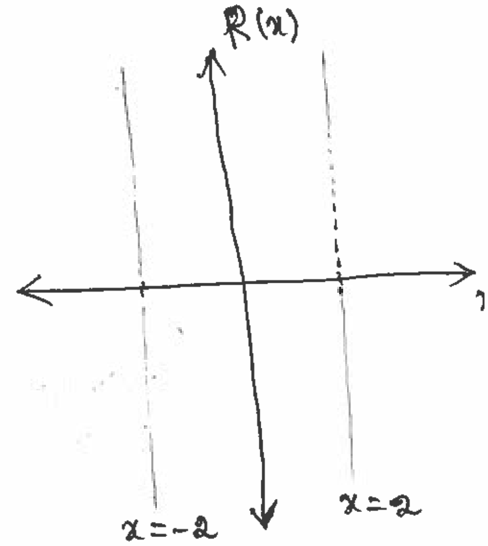
There is no solution of the above equation. Therefore, there is no x-intercept of the graph of $R(x)$:

Step-4

Vertical asymptotes:

$(x+2)(x-2)=0$

Either $x=2$ or $x=-2$.



Step-5

Horizontal asymptote:

Since degree of the numerator is less than the degree of the denominator. The only horizontal asymptote is at $y=0$.

Determine the point of intersection :

$R(x)=0$

$\Rightarrow \dfrac{3}{(x+2)(x-2)} = 0 \Leftarrow$ No solution.

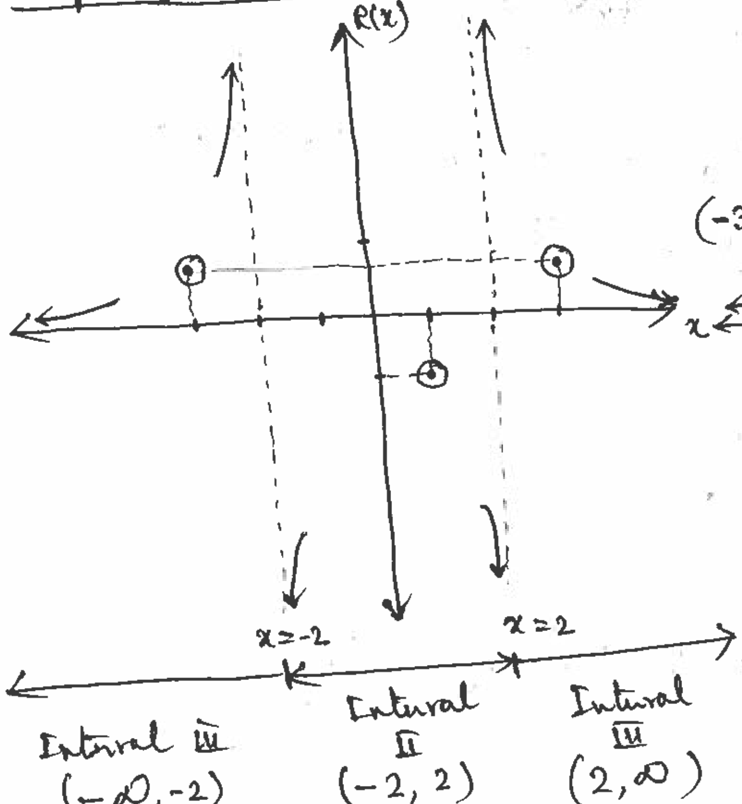
$\therefore$ ~~The~~ The graph does not intersect the x-axis.

Step-6

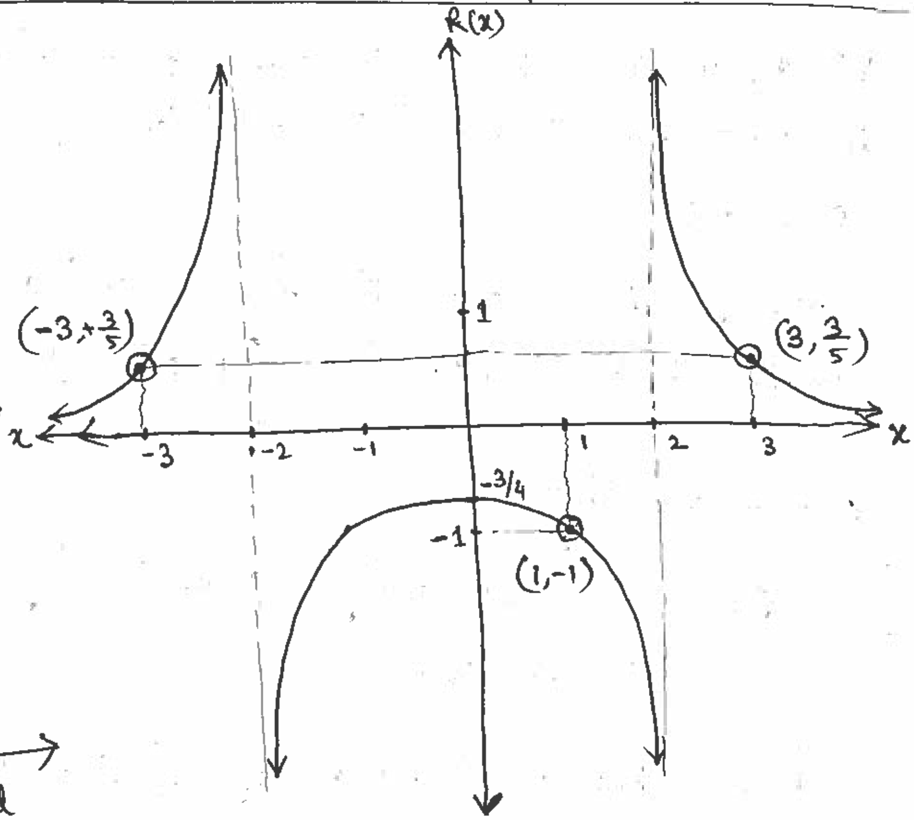
Intervals : $(-\infty,-2)$, $(-2,2)$, $(2,\infty)$

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Number chosen (x)	-3	1	3
Value of $R(x)$	$3/5$	-1	$3/5$
Location of graph	Above x-axis	Below x-axis	Above x-axis
Point on the graph	$(-3, 3/5)$	$(1, -1)$	$(3, \frac{3}{5})$

Step-7 and Step-8



Step-7



Step-8

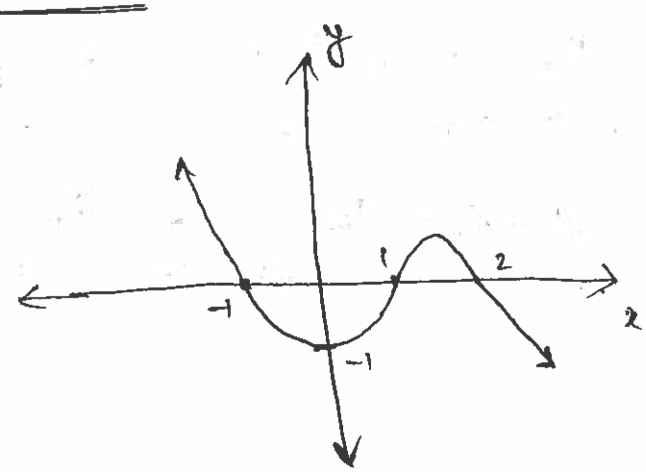
5.4 Polynomial and Rational inequalities

Solve polynomial equations

Graphically

Problem

6. Use the graph of the function to solve (a) $f(x) < 0$ (b) $f(x) \geq 0$



(a) Determine the intervals of x for which the graph is ~~above~~ below the x-axis
From the graph, we can see that $f(x) < 0$ for $\underline{-1 < x < 1}$ and $\underline{x > 2}$.

The solution set is therefore $\{x \mid -1 < x < 1 \text{ or } x > 2\}$, or using interval notation, $\underline{(-1,1) \cup (2,\infty)}$.

(b) Determine the intervals of x for which the graph is above the x-axis. We see that $f(x) \geq 0$ for $\underline{1 \leq x \leq 2}$ and $\underline{x \leq -1}$.

The solution set is therefore $\{x \mid 1 \leq x \leq 2 \text{ or } x \leq -1\}$, or in the interval notation, $\underline{[1,2] \cup (-\infty,-1]}$.

<u>Algebraically</u>.

<u>21</u>. Solve the inequality algebraically.

$$x^3 - 4x^2 > 0$$

Find the real zeros of $f(x) = x^3 - 4x^2$ by solving

$$x^3 - 4x^2 = 0$$

$$\Rightarrow x^2(x-4) = 0$$

Either, $x = 0$ or, $x - 4 = 0$

$$\Rightarrow x = 4.$$

Use the real zeros to separate the real number line into three intervals: $(-\infty, 0)$, $(0,4)$, $(4,\infty)$.

Select a test number in each interval and evaluate $f(x) = x^3 - 4x^2$ at each number to determine if $f(x)$ is positive or negative.

:

Interval	$(-\infty,0)$	$(0,4)$	$(4,\infty)$
Number chosen	-1	2	5
value of $f(x)$	-5	-8	25
conclusion	Negative	Negative	Positive

Since we want to know where $f(x)$ is positive, we conclude that $f(x)>0$ for all numbers x for which $x>4$. The solution set is therefore $\{x \mid x>4\}$ or in the interval notation $(4,\infty)$.

Solve Rational inequality

Graphically

16. Solve the inequality by using the graph of the function.

$$R(x) = \frac{x}{(x-1)(x+2)} < 0.$$

See the graph of the function $R(x)$ in problem number 8 in 5.3.

The value of the function $R(x)$ is negative in the intervals $(-\infty,-2)$ and $(0,1)$. The solution is therefor $R(x)<0$ for $\{x \mid -\infty < x < -2$ and $0<x<1\}$ or in the interval notation $(-\infty,-2) \cup (0,1)$.

Algebraically

40. Solve the inequality algebraically.

$$\frac{x+2}{x-4} \geq 1$$

$$\Rightarrow \frac{x+2}{x-4} - 1 \geq 0$$

$$\Rightarrow \frac{\cancel{x}+2-\cancel{x}+4}{x-4} \geq 0$$

$$\Rightarrow \frac{6}{x-4} \geq 0$$

The zeros of the function can be found by solving

$$f(x) = 0$$

$\Rightarrow \frac{6}{x-4} = 0$, which can not be solved.

$\therefore$ There is no zero of the function.

The function is undefined for $x = 4$.

$\therefore$ Separate the real number line into two intervals:

$(-\infty, 4)$ and $(4, \infty)$.

Interval	$(-\infty, 4)$	$(4, \infty)$
Number chosen	2	5
value of $f(x)$	-3	6
Conclusion	Negative	positive.

Since we need to know the interval, where the function is greater or equals 0, the solution is $f(x) \geq 0$ for $\{x \mid 4 \leq x \leq \infty\}$ or in the interval notation $[4, \infty)$.