# Lecture-7 February 18. 2015

## 4.1 Linear functions:

In lecture-2, we discussed about linear equations. A linear equation is an equation, where each term of the equation is either a constant or a variable of power one multiplied with a constant or a variable of power one only.

Examples: $y = mx + c$ , $m, c$ = constants, $x, y$ = variables

$5x + 6y = 4$ , $x, y$ = variables.

Quite similarly, a linear function is a function, each term of which is of power one of the variable.

Example : $y = f(x) = mx + b$ , $m, b$ = Constants, $x \to$ variable.

## 1. Graph linear functions :

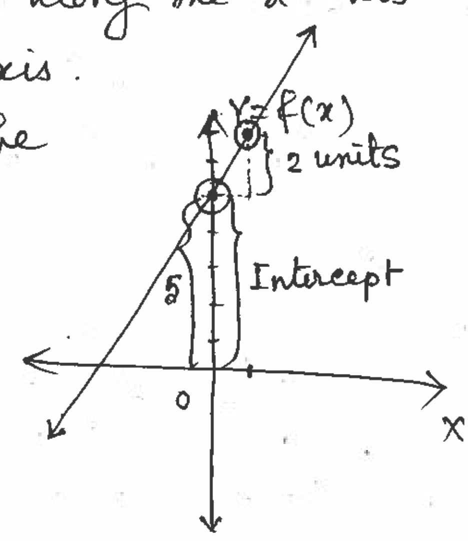
Graphing a linear function of the form $y = f(x) = mx + b$ is quite easy work. As you know, the equation is a linear function in the slope-intercept form, where 'b' is the intercept at the y-axis and 'm' is the slope of the function. Let us see one example :

$$f(x) = 2x + 5$$

$$\text{Slope} = \frac{\text{Change in } y\text{-coordinate}}{\text{Change in } x\text{- coordinate}} = \frac{\Delta y}{\Delta x} = 2$$

This means, one needs to walk one unit along the x-axis first and then 2 units along the positive y axis.

As the function intersects at $y = 5$ on the y-axis, the graph of the function can be plotted easily.



2. A linear function can be identified from another simple trick. Consider the previous example:

$$f(x) = 2x+5$$

Look at the table, where the values of $f(x)$ have been shown for the corresponding values of $x$.

| $f(x)$ | 7 | 9 | 11 | 13 | 3 | 1 | -1 |
|---|---|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | -1 | -2 | -3 |

If the average rate of growth or decay of $f(x)$ is a constant with the constant growth or decay of $x$, the function $f(x)$ is called linear.

3. Increasing or decreasing linear functions:

A linear function $y = f(x) = mx+b$ is increasing over its domain if the slope $m$ is positive. Therefore, our example: $y = f(x) = 2x+5$ is an example of increasing function. On the other hand, $f(x) = -3x+7$ is a decreasing function because its slope is $(-3)$, which is negative. Thus, a linear function is a constant linear function over its domain, if its slope $m = 0$.

4. Build linear models from verbal descriptions:

If the average rate of change of a function is a constant $m$, a linear function $f$ can be used to build the relation between two variables as follows

$$f(x) = mx+b,$$

where '$b$' is the value of $f$ at 0, i.e. $b = f(0)$.

Problems

30. Suppose that $f(x) = 3x+5$ and $g(x) = -2x+15$.
(a) Solve $f(x) = 0$ (b) Solve $f(x) < 0$ (c) Solve $f(x) = g(x)$
(d) Solve $f(x) \geq g(x)$ (e) Graph $y = f(x)$ and $y = g(x)$ and label the point that represents the solution to the equation $f(x) = g(x)$.

(a) $f(x) = 0$

$\Rightarrow 3x + 5 = 0$

$\Rightarrow 3x = -5$

$\Rightarrow x = -\frac{5}{3}$

(b) $f(x) < 0$

$\Rightarrow 3x + 5 < 0$

$\Rightarrow 3x < -5$

$\Rightarrow x < -\frac{5}{3}$

(c) $f(x) = g(x)$

$\Rightarrow 3x + 5 = -2x + 15$

$\Rightarrow 3x + 2x = 15 - 5$

$\Rightarrow 5x = 10$

$\Rightarrow x = \frac{10}{5} = 2$

(d) $f(x) \geq g(x)$

$\Rightarrow 3x + 5 \geq -2x + 15$

$\Rightarrow 3x + 2x \geq 15 - 5$

$\Rightarrow 5x \geq 10$

$\Rightarrow x \geq \frac{10}{5}$

$\Rightarrow x \geq 2$

(e)

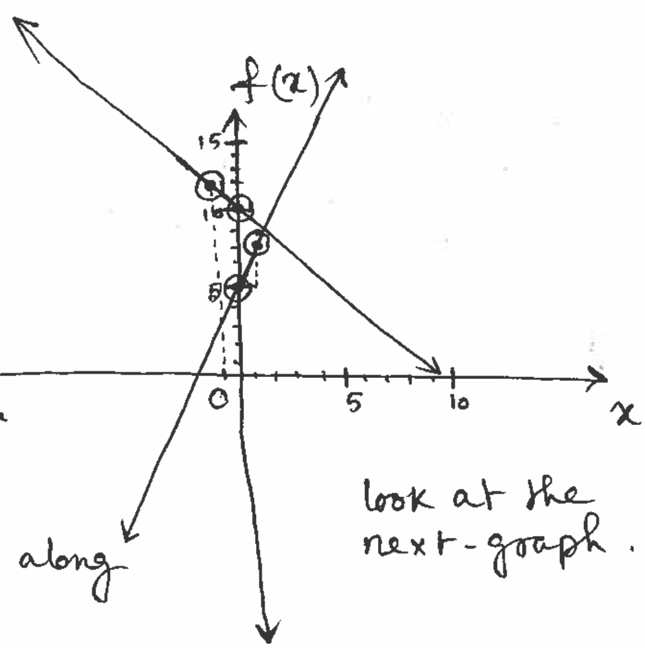
$f(x) = 3x + 5$

intercept at y axis = 5

slope $= \frac{\Delta y}{\Delta x} = \frac{\text{Change in y-co-ord}}{\text{Change in x-coordinate}}$

$= \frac{3}{1}$

That means, if one walks 1 unit along




look at the next-graph.

the positive x-axis, one also needs to walk 3 units along positive y-axis to reach the graph.

Also look at the slanting of the graph as its slope is positive.

Similarly for $g(x) = -2x + 15$

$m = \text{slope} = \dfrac{-2}{1}$, intercept $= 15$ at (+ve) y-axis.

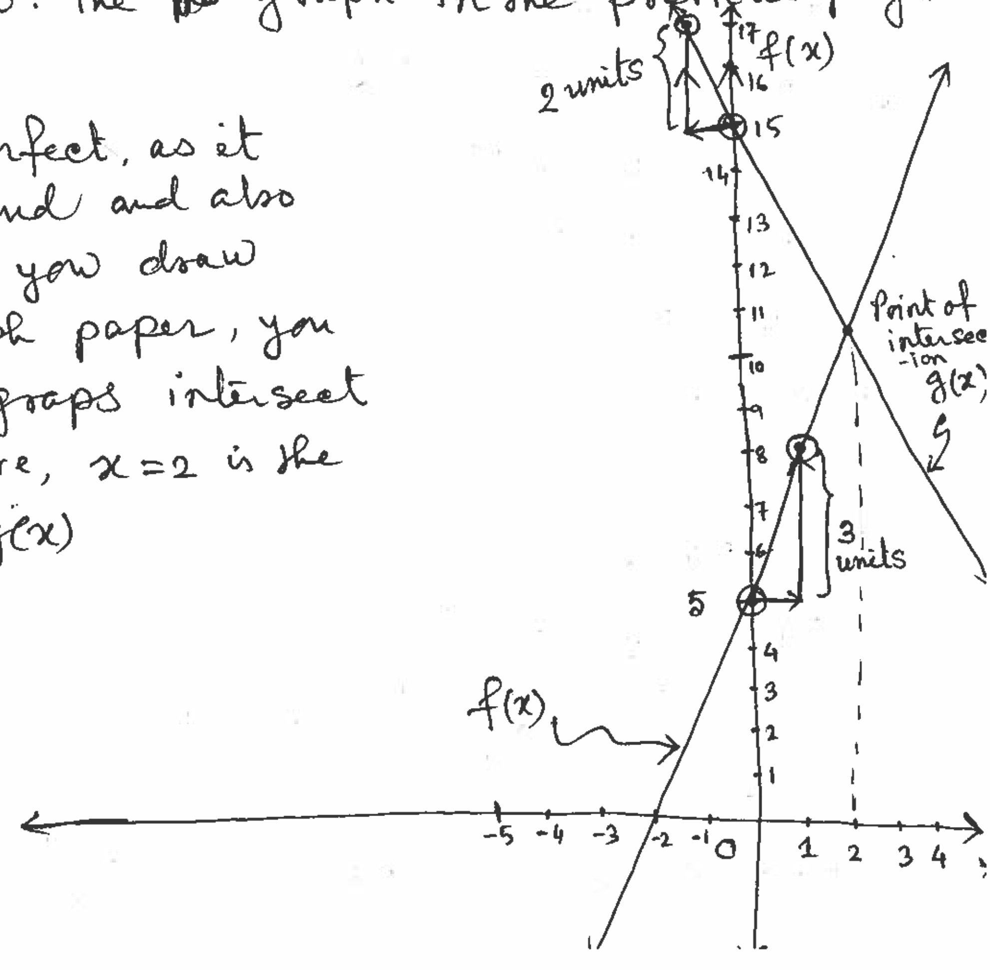
In this case slope is (-ve) and the ~~stat~~ graph slants correspondingly. However, there are two options to draw the graph as follows —

1. $m = \dfrac{-2}{1} = \dfrac{\Delta y}{\Delta x}$ $\Rightarrow$ walk 1 unit along (+ve) x-axis and then walk 2 units along (-ve) y axis.

2. $m = \dfrac{2}{-1} = \dfrac{\Delta y}{\Delta x}$ $\Rightarrow$ Walk 1 unit along (-ve) x-axis and then walk 2 units along (+ve) y-axis.

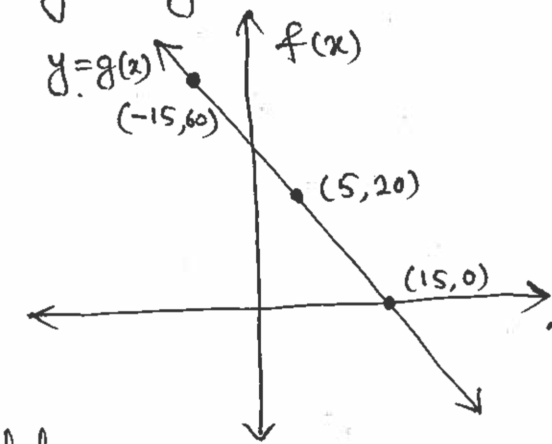
We will take the 2nd option. Of course, taking option-1, the picture does not change. Have a look at the following graph for bigger view. The ~~gr~~ graph in the previous page shows the same.

The graph is not perfect, as it has been drawn by hand and also on plane paper. If you draw the graph on a graph paper, you can see that the graps intersect at $x = 2$. Therefore, $x=2$ is the solution for $f(x) = g(x)$

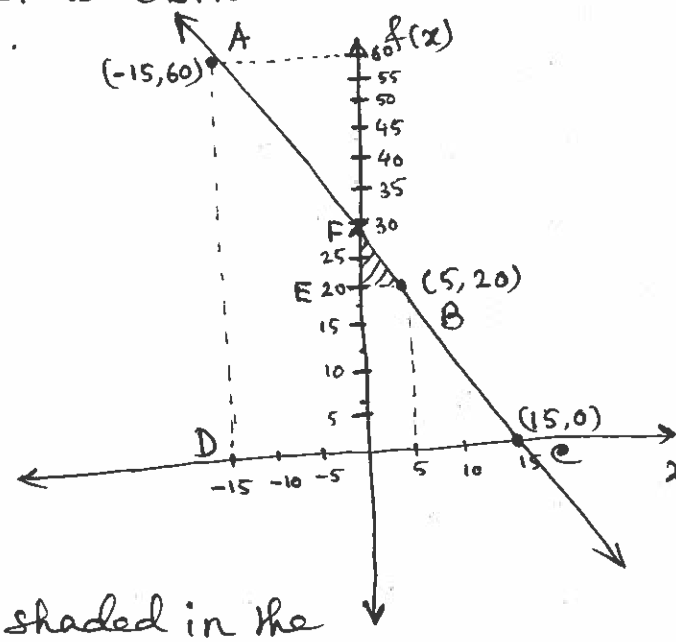


32. In parts (a) - (f), use the following figure.

(a) Solve $g(x) = 20$
(b) Solve $g(x) = 60$
(c) Solve $g(x) = 0$
(d) Solve $g(x) > 20$
(e) Solve $g(x) \leq 60$
(f) Solve $0 < g(x) < 60$




To answer (a) - (f), we need to build a linear function of the graph given in the figure. That means, we need to find the slope (m) and intercept to obtain a linear function. It is obvious that the graph has a negative slope.




It is easy to find the slope of the graph –

$$m = \frac{\Delta y}{\Delta x} = \frac{\overline{AD}}{\overline{DC}}$$

$$= \frac{60}{-30} = -2$$

Now, if you look at the small triangle $\triangleleft BEF$, which has been shaded in the picture, you will find the intercept.

From the given data, you can say $\overline{BE} = 5$. Therefore, $\overline{EF}$ has to be 10, so that slope $\left(m = \frac{10}{-5} = -2\right)$ becomes $(-2)$.

$\therefore$ Intercept at the y-axis $= (20 + 10) = 30$.

$\therefore$ The linear function is $g(x) = -2x + 30$

Now you can answer (a)-(f) very easily.

(a) $g(x) = 20$

$\Rightarrow -2x+30 = 20$

$\Rightarrow -2x = 20-30$

$\Rightarrow -2x = -10$

$\Rightarrow x = \frac{-10}{-2} = 5$

(b) $g(x) = 60$

$\Rightarrow -2x+30 = 60$

$\Rightarrow -2x = 60-30$

$\Rightarrow -2x = 30$

$\Rightarrow x = \frac{30}{-2} = -15$

(c) $g(x) = 0$

$\Rightarrow -2x+30 = 0$

$\Rightarrow -2x = -30$

$\Rightarrow x = \frac{-30}{-2} = 15$

(d) $g(x) > 20$

$\Rightarrow -2x+30 > 20$

$\Rightarrow -2x > 20-30$

$\Rightarrow -2x > -10$

$\Rightarrow \frac{-2x}{-2} < \frac{-10}{-2}$

$\Rightarrow x < 5$

(e) $g(x) \leq 60$

$\Rightarrow -2x+30 \leq 60$

$\Rightarrow -2x \leq 60-30$

$\Rightarrow -2x \leq 30$

$\Rightarrow \frac{-2x}{-2} \geq \frac{30}{-2}$

$\Rightarrow x \geq -15$

(f) $0 < g(x) < 60$

$\Rightarrow 0 < -2x+30 < 60$

$\Rightarrow -30 < -2x < 60-30$

$\Rightarrow -30 < -2x < 30$

$\Rightarrow \frac{-30}{-2} > x > \frac{30}{-2}$

$\Rightarrow 15 > x > -15$

40. Suppose that the quantity supplied S and quantity demanded D of hot dogs at a baseball game are given by the following functions:

$$S(p) = -2000 + 3000p$$
$$D(p) = 10000 - 1000p$$

where p is the price of a hot dog.

(a) Find the equilibrium price for hot dogs at the baseball game. what is the equilibrium quantity?

(b) Determine the price for which quantity demanded is less than quantity supplied?

(c) what do you think will eventually happen to the price of hot dogs if quantity demanded is less than quantity supplied.

To answer the above questions, one needs to know the terminologies as written below —

(i) Quantity supplied: Amount of goods of a product that a company is willing to make available for sale at a given price.

(ii) Quantity demanded: Amount of goods of a product that consumers are willing to purchase at a given price.

(iii) Equilibrium price: The price at which quantity supplied equals quantity demanded. i.e. the price at which $S(p) = D(p)$.

(a) $S(p) = D(p)$

$\Rightarrow -2000 + 3000p = 10000 - 1000p$

$\Rightarrow 3000p + 1000p = 10000 + 2000$

$\Rightarrow 4000p = 12000$

$\Rightarrow p = \dfrac{12000}{4000} = 3$

Equilibrium price $= 3$ unit.

To find the equilibrium quantity, one needs to put the value of $p$ to any one of the equations $S(p) = -2000 + 3000p$ and $D(p) = 10000 - 1000p$. Let us consider the first one –

$$S(p) = -2000 + 3000p$$
$$= -2000 + 3000 \times 3$$
$$= -2000 + 9000$$
$$= 7000$$

$\therefore$ Equilibrium quantity $= 7000$.

(b) Quantity demanded $<$ Quantity supplied.

$$\Rightarrow \quad D(p) < S(p)$$
$$\Rightarrow \quad 10000 - 1000p < -2000 + 3000p$$
$$\Rightarrow \quad -1000p - 3000p < -2000 - 10000$$
$$\Rightarrow \quad -4000p < -12000$$
$$\Rightarrow \quad p > 3$$

(c) The price of hot dogs will eventually go higher than 3.

49. A truck rental company rents a truck for one day by charging \$29 plus \$0.07 per mile.
(a) Write a linear model that relates the cost $C$, in dollars, of renting the truck to the number $x$ of miles driven.
(b) What is the cost of renting the truck if the truck is driven 110 miles? 230 miles?

(a) $C(x) = $ (price per mile) $\times$ (Number of miles driven) $+ C(0)$, which is the price for driving 0 miles.

$\therefore c(x) = 0.07x + 29$

(b) For 110 miles –

$$\text{cost} = c(x) = 0.07 \times 110 + 29 \ \$$$

For 230 miles –

$$\text{Cost} = c(x) = 0.07 \times 230 + 29 \ . \$$$

## 4.3 Quadratic functions:

### 1. Graph a quadratic function using transformations

A quadratic function is of the form

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\frac{b^2}{4a^2} \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \\
&= a(x-h)^2 + K,
\end{aligned}
$$

where, $h = -\frac{b}{2a}$ and $K = \frac{4ac - b^2}{4a}$.

Obviously, the graph of $f(x) = a(x-h)^2 + K$ is the parabola $y = ax^2$ shifted horizontally $h$ units (replaces $x$ by $x-h$) and vertically $K$-units (add $K$). As a result the vertex is at $(h, K)$, and the graph opens up ~~at~~ if $a > 0$ and down if $a < 0$. The axis of symmetry is the vertical line $x = h$.

Problem

30. Graph the function $f(x) = \frac{2}{3}x^2 + \frac{4}{3}x - 1$ by starting with the graph of $y = x^2$ and using transformations (shifting, compressing, streatching and/or reflection).

$$f(x) = \frac{2}{3}x^2 + \frac{4}{3}x - 1$$

$$= \frac{2}{3}(x^2 + 2x) - 1$$

$$= \frac{2}{3}(x^2 + 2\cdot x + 1) - 1 - 1$$

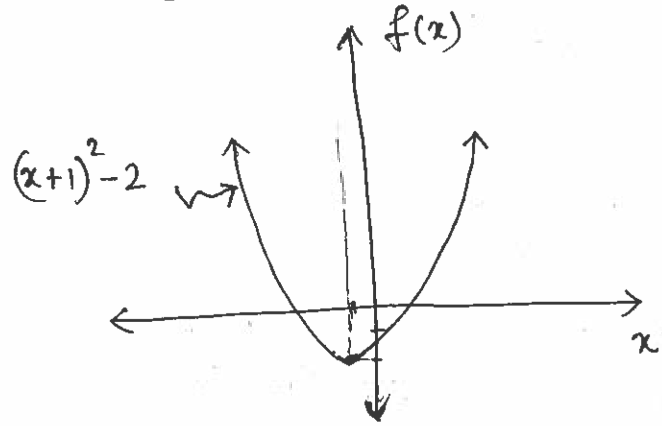
$$= \frac{2}{3}(x+1)^2 - 2$$

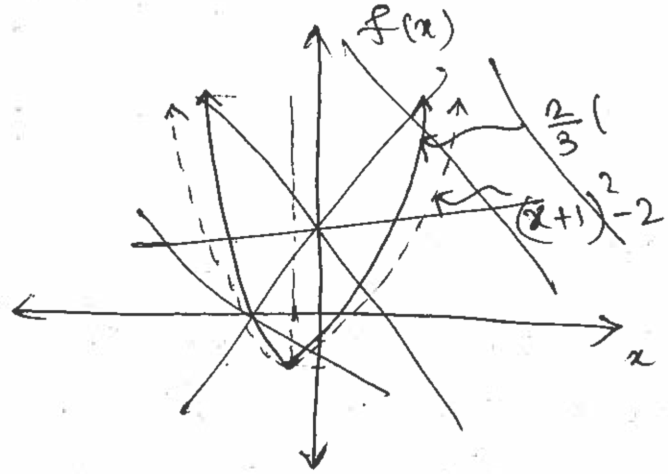
Compare this function with the standard one

$$f(x) = a(x-h)^2 + k,$$

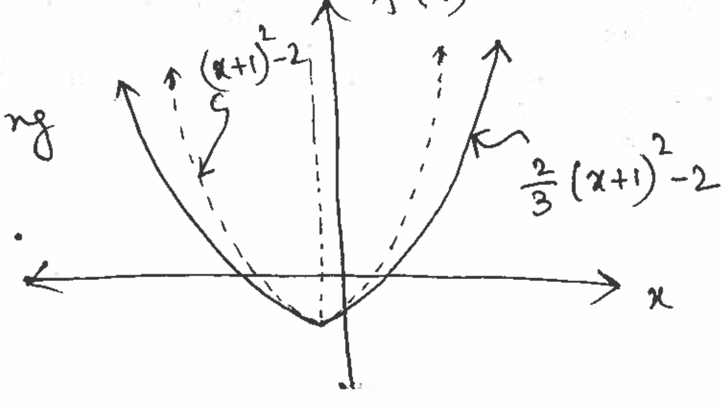
which says that $a = \frac{2}{3}$, $(h,k) = (-1,-2)$

$\therefore$ The vertex of the ~~graph~~ parabola is at $(-1,-2)$ and as $a = \frac{2}{3} > 0$, the graph opens up.




Remember, from the previous class, as the graph is multiplied by a factor of $a = \frac{2}{3}$ which is in between 0 and 1, the resulting graph will be compressed along y axis and. y-coordinate will be multiplied by $\frac{2}{3}$ to obtain the final graph.



2. <u>Identify the vertex and axis of symmetry of a quadratic function:</u>

In the previous section, we took the quadratic function $f(x) = ax^2+bx+c$ and completed the square to obtain the vertex of the graph. The new function looked like

$$f(x) = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

Actually, one does not need so much of work, one can remember the x-coordinate of the function, which is $\left(-\frac{b}{2a}\right)$. The y-coordinate can be then be found by evaluating f at $\left(-\frac{b}{2a}\right)$. i.e.

$$\begin{cases} h = \frac{-b}{2a} \\ k = f\left(-\frac{b}{2a}\right) \end{cases}$$

$\therefore$ The vertex is at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

axis of symmetry: the line $x = -\frac{b}{2a}$.

3. <u>Graph a quadratic function using its vertex, axis and intercepts.</u>

The y-intercept of the function is found by putting $x=0$ to the equation, which gives

$$\begin{aligned} f(x) &= ax^2+bx+c \\ &= a\cdot(0)^2 + b\cdot 0 + c \\ &= c \end{aligned}$$

$\therefore$ The y-intercept is at c

To find x-intercept/intercepts, one needs to solve the equation by quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

(a) Now look at the discriminant $(b^2-4ac)$. If the discriminant $(b^2-4ac) > 0$, the graph has two distinct $x$-intercepts, so it ~~crossesed~~ crosses the $x$-axis in two places.

(b) If $(b^2-4ac) = 0$, the graph has only one $x$-intercept. So, it touches the $x$-axis at its vertex.

(c) If $(b^2-4ac) < 0$, the graph has no $x$-intercepts. So it does not cross or touch the $x$-axis.

Problems

41. (a) Graph the quadratic function by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, $y$-intercept and $x$-intercepts.
(b) Determine the domain and range of the function.
(c) Determine ~~whether~~ where the function is increasing and where it is decreasing. $f(x) = -2x^2+2x-3$

$$f(x) = -2x^2+2x-3 \equiv ax^2+bx+c$$

The $x$-coordinate of the vertex of the function is $\left(-\frac{b}{2a}\right)$.

$\therefore$ $x$-coordinate $= -\frac{b}{2a} = -\frac{\cancel{2}}{\cancel{2}\cdot(-2)} = \frac{1}{2}$

$\therefore$ $y$-coordinate is therefore $f\left(\frac{1}{2}\right) = -2\cdot\left(\frac{1}{2}\right)^2 + \cancel{2}\cdot\frac{1}{\cancel{2}} - 3$

$$= -2\cdot\frac{1}{4}+1-3$$

$$= -\frac{1}{2}-2$$

$$= -\frac{5}{2}.$$

$\therefore$ Vertex $= \left(\frac{1}{2}, -\frac{5}{2}\right)$

Axis of symmetry is $x = \frac{1}{2}$.

$y$ intercept is $(-3)$

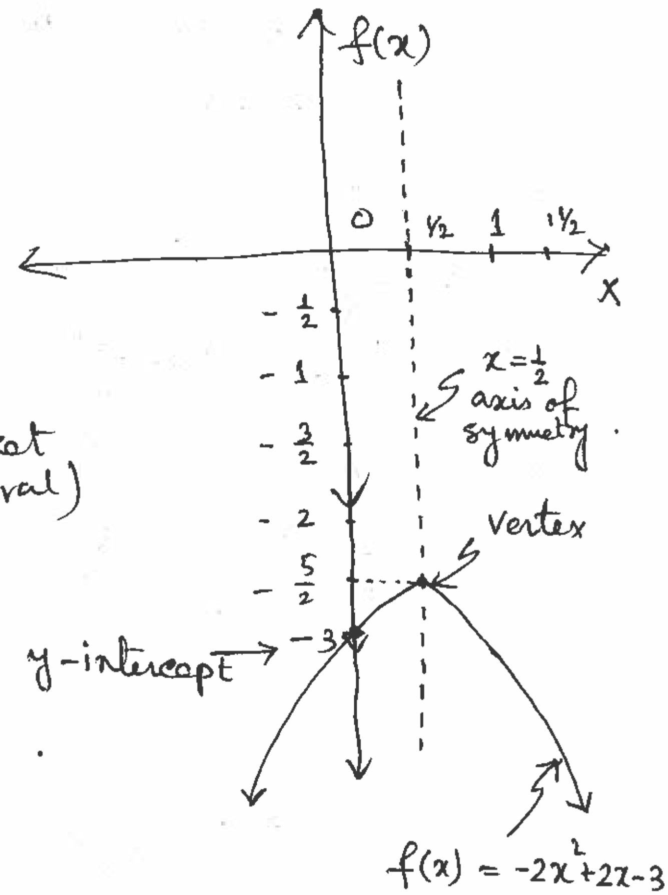
Now apply quadratic formula to solve $f(x) = -2x^2+2x-3$. The discriminant is $(b^2-4ac) = (2)^2 - 4\cdot(-2)(-3)$

$$= 4 - 24$$

$$= -20$$

As the discriminant is negative, the graph does not have $x$-intercepts.

Assembling all the above information, we now graph the function




(b) The domain of $f$ is the set of all real numbers. Based on the graph the range of $f$ is the interval $[-\frac{5}{2}, -\infty)$

Square bracket (closed interval) Round bracket (open interval)

(c) The function $f$ is decreasing on the interval $(-\infty, \frac{1}{2})$ and increasing on the interval $(\frac{1}{2}, \infty)$.

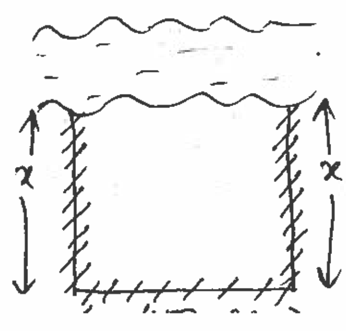
## 4.4 Quadratic models:

9. Problem

A farmer with 4000 meters of fencing wants to enclose a rectangular plot that borders on a river. If the farmer does not fence the side along the river, what is the largest area that can be enclosed?

$\therefore$ Area of the plot is $= x(4000-2x)$ unit$^2$

$$= -2x^2 + 4000x \text{ unit}^2$$



$\therefore f(x) = -2x^2 + 4000x + 0 \equiv ax^2 + bx + c$

$$\begin{cases} a = -2 \\ b = 4000 \\ c = 0 \end{cases}$$

$\therefore$ Vertex $= \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(\frac{-4000}{2\cdot(-2)}, f\left(\frac{-4000}{2(-2)}\right)\right)$

$= \left(1000, f(1000)\right) = (1000, 2000000)$

$f(1000) = -2\cdot(1000)^2 + 4000 \cdot 1000$

$= -2000000 + 4000000$

$= 2000000$

axis of symmetry : $x = 1000$

y-intercept is at 0.

$(b^2 - 4ac) = (4000)^2 - 4\cdot(-2)\cdot 0 = 16000000 = (+ve)$

$\therefore$ There are two x-intercepts.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4000 \pm \sqrt{16000000}}{2(-2)}$$

$$= \frac{-4000 \pm 4000}{-4}$$

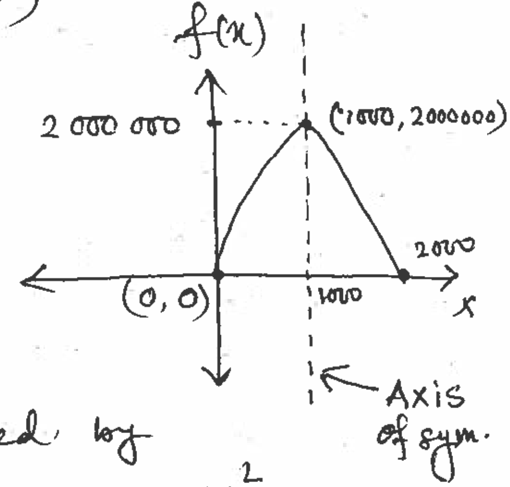
$\therefore$ The x-intercepts are at $(0, 2000)$

$\therefore$ The maximum value of $x$ is at

$x = 1000$

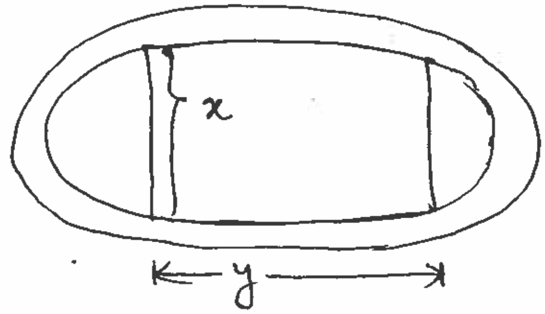
The maximum value of the area is therefore

$f(1000) = 2000000$ unit$^2$




$\therefore$ The largest rectangle that can be enclosed by 4000 meters of fence has an area of 2000000 $m^2$.

17. A track and field playing area is in the shape of a rectangle with semicircles at each end. The inside perimeter of the track is to be 1500 meters. What should be the dimensions of the rectangle be so that the area of the rectangle is maximum.




Each semicircle has the perimeter $= \pi x$

$\therefore$ The inside perimeter of the track $= \pi x + \pi x + 2y$

$= 2\pi x + 2y$

$\therefore 2\pi x + 2y = 1500$

$\Rightarrow 2y = 1500 - 2\pi x$

$\Rightarrow y = 750 - \pi x$

$\therefore$ The area of the rectangle $= x \cdot y \; m^2$

$A(x) = x \cdot (750 - \pi x) \; m^2$

$A(x) = -\pi x^2 + 750\, x \; m^2$

Compare $A(x)$ with the function $ax^2 + bx + c$

$\therefore \begin{cases} a = -\pi \\ b = 750 \\ c = 0 \end{cases}$

$\therefore$ vertex is at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{750}{2(-\pi)}, f\left(\frac{-750}{2(-\pi)}\right)\right)$

$= \left(\frac{375}{\pi}, f\left(\frac{375}{\pi}\right)\right)$

$= \left(\frac{375}{\pi}, \frac{(375)^2}{\pi}\right)$

$f\left(\frac{375}{\pi}\right) = -\pi\left(\frac{375}{\pi}\right)^2 + 750 \cdot \frac{375}{\pi}$

$= \frac{-(375)^2}{\pi} + \frac{750 \cdot 375}{\pi}$

$= \frac{(375)^2}{\pi}$

axis of symmetry : $x = \frac{375}{\pi}$

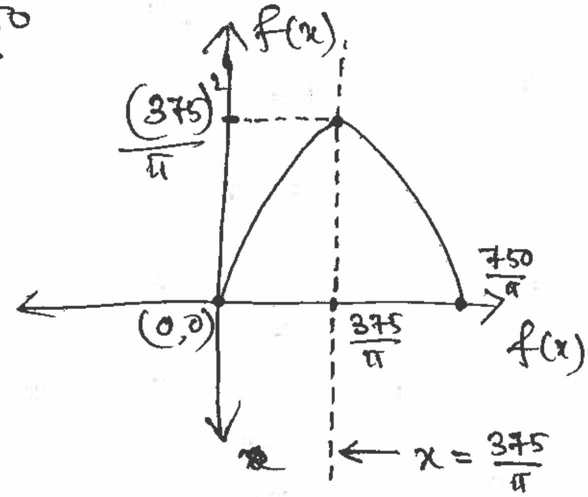
$y$-intercept $= 0$

Discriminant $= b^2 - 4ac = (750)^2 - 4\cdot(-\pi)\cdot 0$

$= (750)^2$

$\therefore \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-750 \pm \sqrt{(750)^2}}{2\cdot(-\pi)}$

$= \frac{-750 \pm 750}{-2\pi}$




$\therefore$ $x$-intercepts are at $\left(0, \frac{750}{\pi}\right)$

$\therefore$ The maximum value of $x$

$= \frac{375}{\pi}$.

$\therefore$ The maximum value of the area of the rectangle is

$$f\left(\frac{375}{\pi}\right) = \frac{(375)^2}{\pi} \text{ m}^2 .$$

$\therefore$ Dimension of the rectange where the area is maximum is —

~~Length~~ Length $= y = 750 - \pi\cdot x$

$= 750 - \cancel{\pi}\cdot \frac{375}{\cancel{\pi}}$

$= 375$ m.

Width $= x = \frac{375}{\pi}$ m.

## 4.5 Inequalities involving quadratic functions :

To solve the inequality

$$ax^2+bx+c > 0 \quad , a \neq 0 ,$$

graph the function $f(x) = ax^2+bx+c$ and from the graph where the graph is above the x-axis, i.e. where $f(x) > 0$.

The values of $x$ for whic $f(x) > 0$ are the solutions of the function $ax^2+bx+c > 0$.

To solve the inequality $ax^2+bx+c < 0$, $a \neq 0$, graph the function $f(x) = ax^2+bx+c$ and determine where the graph is below the x axis. Those are the solutions.

Problems.

8. Solve $x^2+3x-10 > 0$

Graph the function $f(x) = x^2+3x-10$. We know the usual procedure.

$$\text{Vertex} = \left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{3}{2},\ f\left(-\frac{3}{2}\right)\right)$$

$$f\left(-\tfrac{3}{2}\right) = \left(-\tfrac{3}{2}\right)^2 + 3\cdot\left(-\tfrac{3}{2}\right) - 10$$
$$= \frac{9}{4} - \frac{9}{2} - 10$$
$$= \frac{9 - 18 - 40}{4}$$
$$= -\frac{49}{4}$$

$$= \left(-\frac{3}{2},\ -\frac{49}{4}\right)$$

axis of symetry: $x = -\frac{3}{2}$.

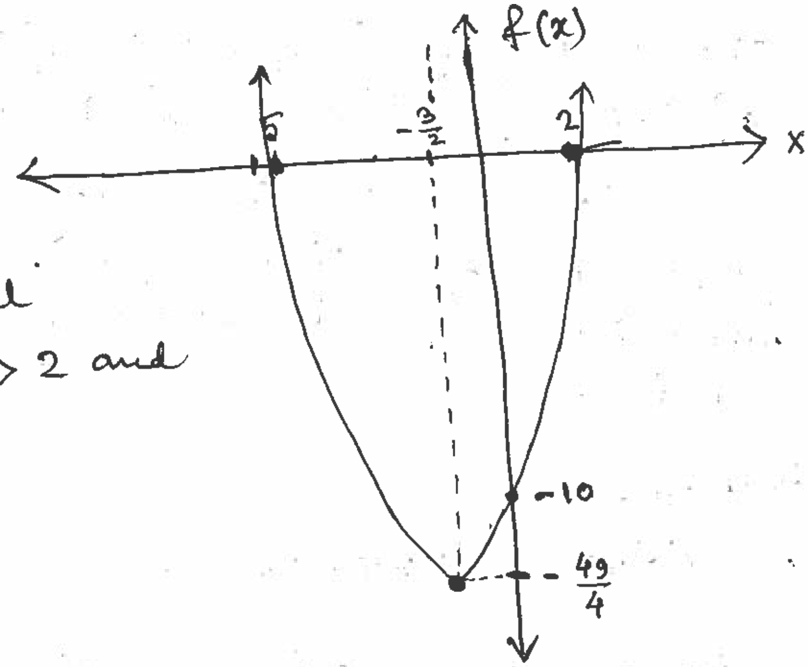
y intercept is at $(-10)$.

Discriminant $b^2-4ac = 3^2 - 4\cdot1\cdot(-10)$
$= 9 + 40 = 49 = (+ve)$

$$\therefore x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{49}}{2\cdot1} = \frac{-3 \pm 7}{2}$$

$\therefore$ x-intercepts are $\left(\frac{-3+7}{2},\ \frac{-3-7}{2}\right) = (2, -5)$

The graph of f lies above the x-axis for $x > 2$ and $x < -5$. The solution set is therefore all the real numbers where $x > 2$ and $x < -5$.




11. Solve $x^2 - 9 < 0$

Graph the function $f(x) = x^2 - 9$

Comparing with $ax^2 + bx + c$, we have $\begin{cases} a = 1 \\ b = 0 \\ c = -9 \end{cases}$

$\therefore$ Vertex $= \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

$= (0, -9)$

axis of symetry : $x = 0$

y-intercept $= (-9)$

Discriminant $= b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-9) = +ve = 36$

$\therefore x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{0 \pm \sqrt{36}}{2 \cdot 1} = \dfrac{\pm 6}{2}$

$\therefore$ The x-intercepts are $+3$ and $-3$.

The graph f lies below the x-axis for all the real values of x in between 3 and -3. The solution is therefore $-3 < x < 3$ or in the interval notation $(-3, 3)$.