3.3 Properties of Functions:

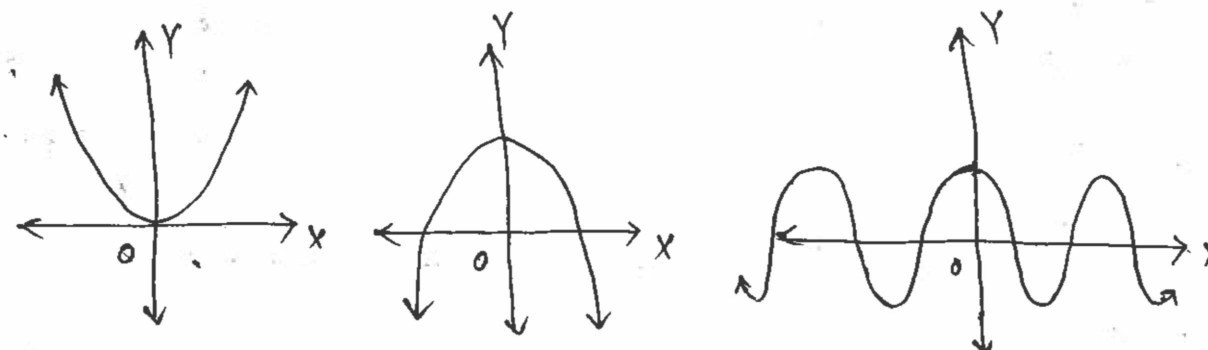
Even and odd functions: A function is said to be an even function if —

$$f(-x) = f(x)$$

Remember, in lecture 4, we tested the symmetry property of a graph. To test the symmetry of the graph with respect to y axis, we replaced x by $(-x)$ and checked whether the equation remained invariant. The above function says that an even function must remain invarient when x is sreplaced by $(-x)$. This means, we have derived another importent property of an even function, which says that an even function must be symmetric with respect to y axis.

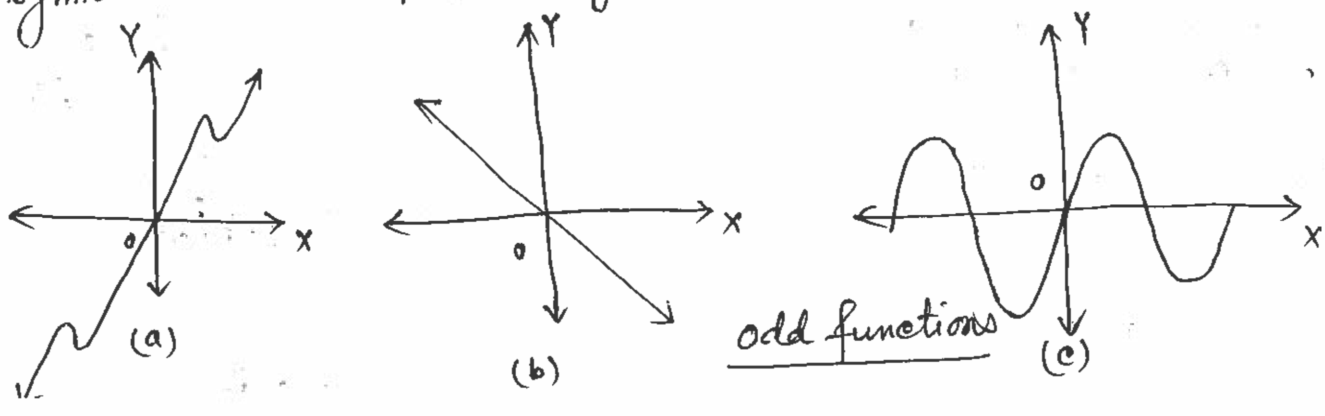
Whereas, for odd functions $f(-x) = -f(x)$, which says a function is said to be odd if and only if its graph is symmetric with respect to the origin.

examples:



(a) (b) (c)

— even functions — because the graphs of the function are symmetric with respect to y-axis.



(a) (b) (c)

odd functions

Problem

Determine algebraically whether each function is even, odd or neither.

34. $f(x) = 2x^4 - x^2$

$$x \to -x \Rightarrow f(-x) = 2(-x)^4 - (-x)^2$$
$$= 2x^4 - x^2 = f(x)$$

$\therefore$ The function $f(x) = 2x^4 - x^2$ is an <u>even function</u>.

43. $h(x) = \dfrac{-x^3}{3x^2-9}$

$$h(-x) = \frac{-(-x)^3}{3(-x)^2-9} = \frac{-(-x^3)}{3x^2-9} = \frac{x^3}{3x^2-9} = -h(x)$$

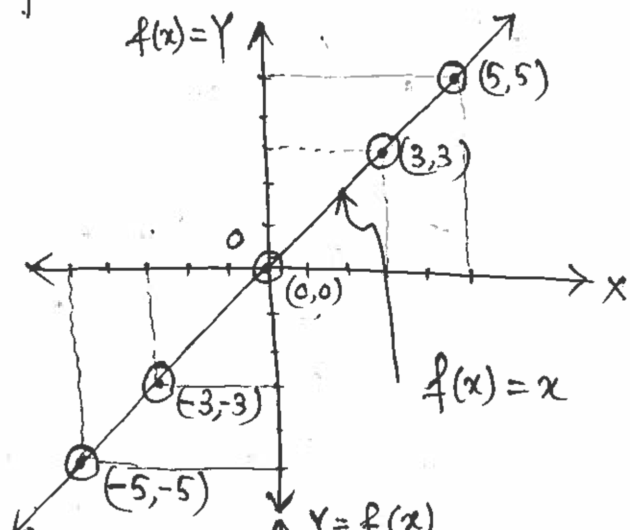
$\therefore$ The function $f(x) = \dfrac{-x^3}{3x^2-9}$ is an <u>odd function</u>.

3.4 Library of functions:

Problems: Sketch the graph of each function. Be sure to label three points on the graph.

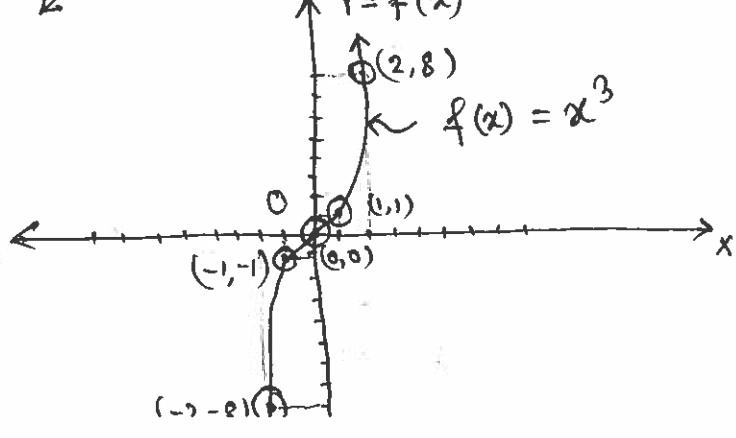
17. $f(x) = x$

x	0	3	5	-3	-5
$f(x)$	0	3	5	-3	-5



19. $f(x) = x^3$

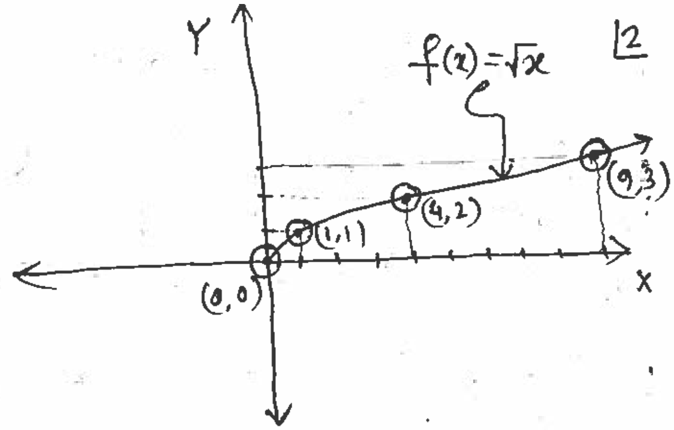
x	0	1	2	-1	-2
$f(x)$	0	1	8	-1	-8



20. $f(x) = \sqrt{x}$

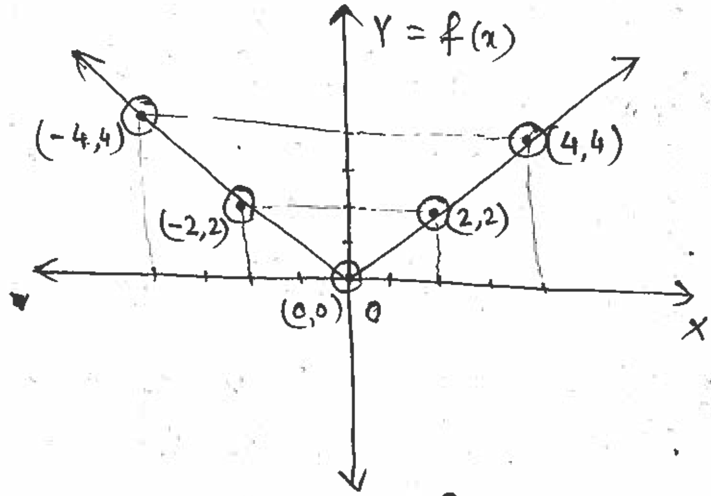
x	0	1	4	9
$f(x)$	0	1	2	3

Negative values of x are not allowed.



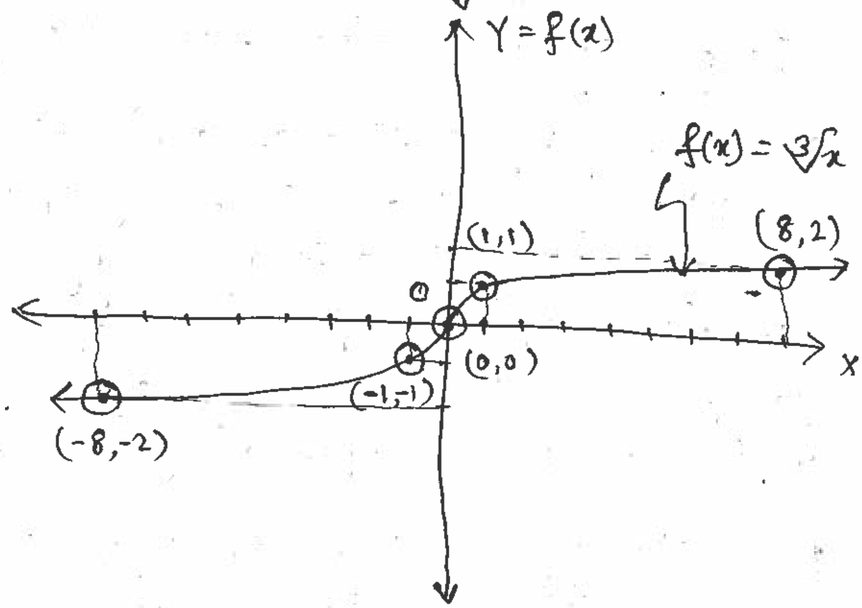
22. $f(x) = |x|$

x	0	2	4	−2	−4
$f(x)$	0	2	4	2	4



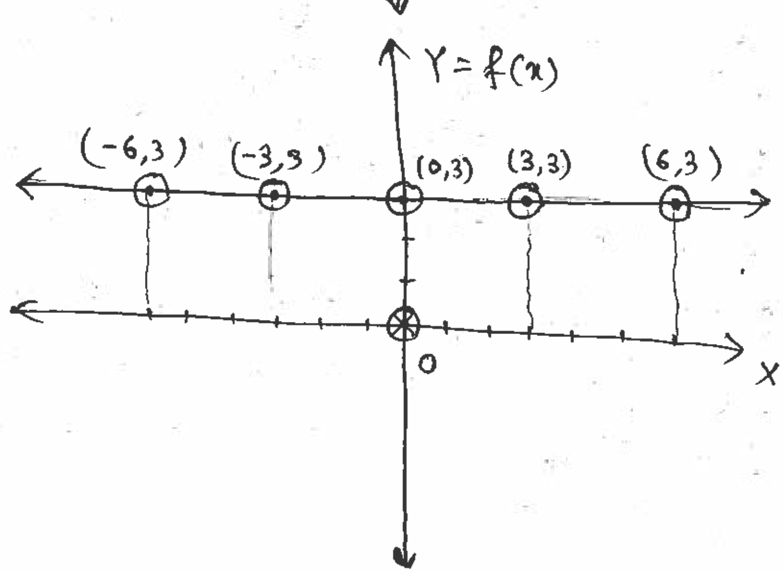
23. $f(x) = \sqrt[3]{x}$

x	0	1	8	−1	−8
$f(x)$	0	1	2	−1	−2



24. $f(x) = 3$

x	0	3	6	−3	−6
$f(x)$	3	3	3	3	3



3.5 Graphing techniques : Transformations

- ① If a positive real number K is added to the output of a function $y = f(x)$, the graph of the new function $y = f(x) + K$ is the graph of f shifted vertically up K-units.
- ② If a positive real number K is subtracted from the output of a function $y = f(x)$, the graph of the new function $y = f(x) - K$ is the graph of f shifted vertically down K units.
- ③ If the argument x of a function f is replaced by $(x-h)$, $h > 0$, the graph of the new function $y = f(x-h)$ is the graph of f shifted horizontally right h units.
- ④ If the argument x of a function f is replaced by $(x+h)$, $h > 0$, the graph of the new function $y = f(x+h)$ is the graph of f shifted horizontally left h units.
- ⑤ When the right side of a function $y = f(x)$ is multiplied by a positive number a, the graph of the new function $y = a\,f(x)$ is obtained by multiplying each y-coordinate on the graph of $y = f(x)$ by a. The new graph is vertically compressed (if $0 < a < 1$) or a vertically stretched (if $a > 1$) version of the graph of $y = f(x)$.
- ⑥ If the argument x of a function $y = f(x)$ is multiplied by a positive number a, the graph of the new function $y = f(ax)$ is obtained by multiplying each x-coordinate of $y = f(x)$ by $\frac{1}{a}$. A horizontal compression results if $a > 1$ and a horizontal stretch occurs if $0 < a < 1$.
- ⑦ When the right side of the function $y = f(x)$ is multiplied by (-1), the graph of the new function $y = -f(x)$ is the reflection about the x-axis of the graph of the function $y = f(x)$.

⑧ When the graph of the function $y = f(x)$ is known, the graph of the new function $y = f(-x)$ is the reflection about the y-axis of the graph of the function $y = f(x)$.

Problems

19. Write the function whose graph is the graph of $y = x^3$, but is shifted to the right 4 units.
Original function : $y = x^3$.
Modified function : shifted to the right by 4 units, according to rule ③

$$y = f(x-4) = (x-4)^3$$

21. Write the function whose graph is the graph of $y = x^3$, but is shifted up 4 units.

Original function : $y = x^3$
Modified function : shifted up by 4 units, according to rule ①

$$y = f(x) + 4 = x^3 + 4$$

23. Reflected about the y-axis.
Original function : $y = f(x) = x^3$
Modified function : Reflected about y-axis, according to the rule ⑧

$$y = f(-x) = (-x)^3 = -x^3$$

25. Vertically stretched by a factor of 4
Original function : $y = f(x) = x^3$
Modified function : Vertically stretched by a factor of 4, according to the rule ⑤

$$y = 4 \cdot f(x) = 4x^3$$

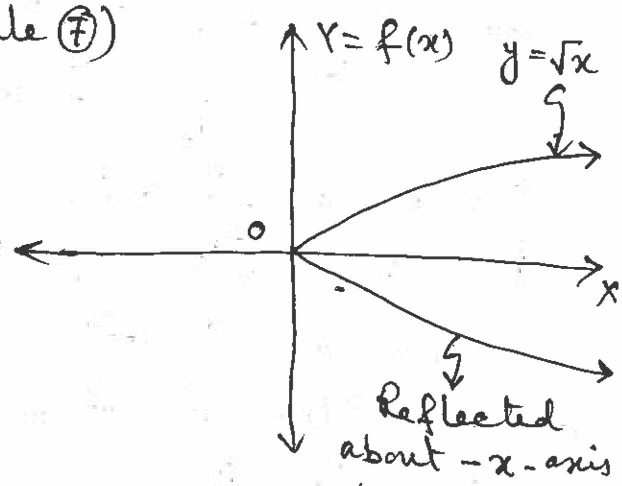
29. Find the function that is finally graphed after each of the following transformations is applied to the graph of $y = \sqrt{x}$ in the order stated.

(1) Reflect about the x-axis
(2) Shift up 2-units.
(3) Shift left 3-units.

Original function: $y = f(x) = \sqrt{x}$

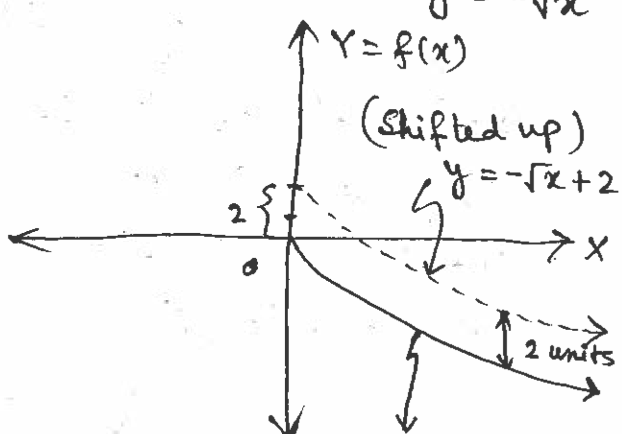
(1) Reflected about the x-axis: (Rule ⑦)

$$y = -f(x) = -\sqrt{x}$$



(2) original function: $y = -f(x) = -\sqrt{x}$
Shift up 2 units: (Rule ①)

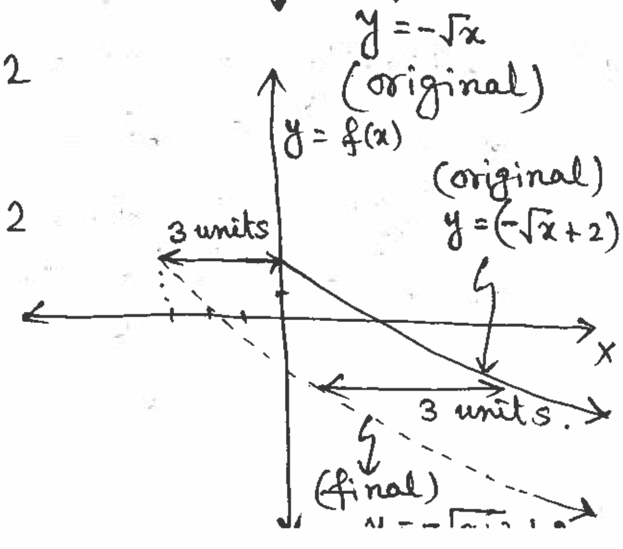
$$y = -f(x) + 2 = -\sqrt{x} + 2$$



(3) Shifted left 3-units.
original function $y = -f(x) + 2 = -\sqrt{x} + 2$
Shifted left 3-units: (Rule ④)

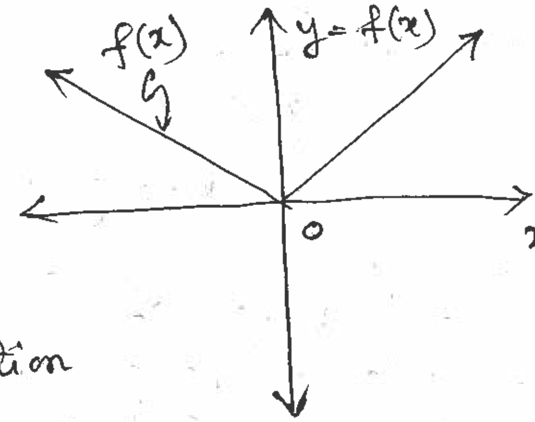
$$y = -f(x+3) + 2 = -\sqrt{x+3} + 2$$

Final function: $\underline{y = -\sqrt{x+3} + 2}$



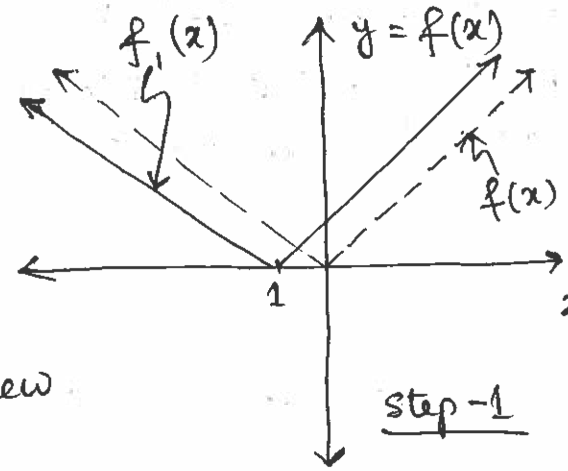
59. Graph the function using the techniques of shifting, compressing, streatching, and/or reflecting. Start with the graph of the basic function and show all stages. Be sure to show at least three key points. Find the domain and the range of the function $g(x) = 2|1-x|$

Let us first consider the basic function: $f(x) = |x|$, and build the function $g(x)$ by using different techniques.



(i) Take a shift to the left by 1 unit. According to the rule ④, the new function becomes:

$$f_1(x) = f(x+1) = |x+1|$$

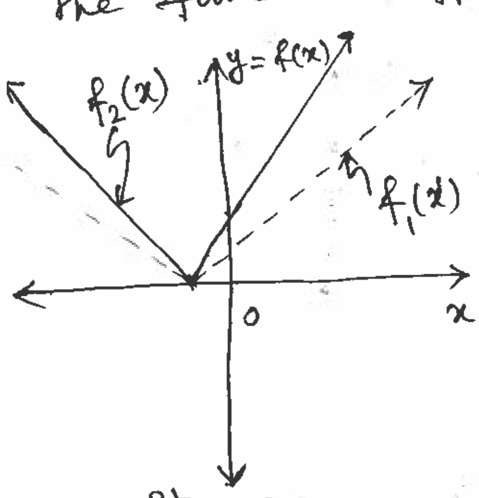


step-1

(ii) At the second step, let us consider rule number ⑤ and multiply $f_1(x)$ by a real positive number 2. The new function, therefore, becomes:

$$f_2(x) = 2f_1(x) = 2|x+1|$$

The function $f_2(x)$ is streached vertically with respect to the function $f_1(x)$



Step-2

(iii) In the last step, we will utilize rule number ⑧ and take a reflection of the function $f_2(x)$ about y-axis. The new function becomes:

$$f_3(x) = f_2(-x) = 2|-x+1| = 2|1-x|$$

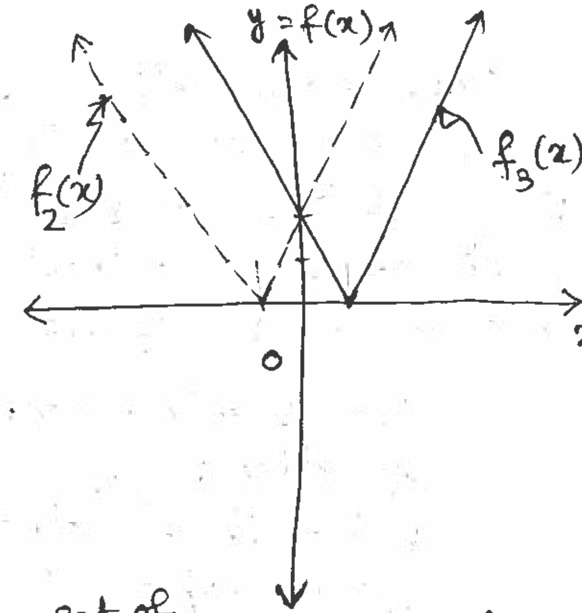
$f_3(x)$ is the final function.

Domain of $g(x) = 2|1-x|$:

The function $g(x)$ is allowed or defined for any value of x on the real line. Therefore, the domain of the function is all real numbers.

$g(x) \in \mathbb{R}$ ← Real number.

↑ Belongs to



Range: Range of function means, the set of values of $g(x)$ in the allowed range of ~~values of~~ x. In our present problem, it is clear that the value of $g(x)$ starts from zero and continues up to positive infinite. The range of the function $g(x)$ is, therefore,

$$g(x) \geq 0$$

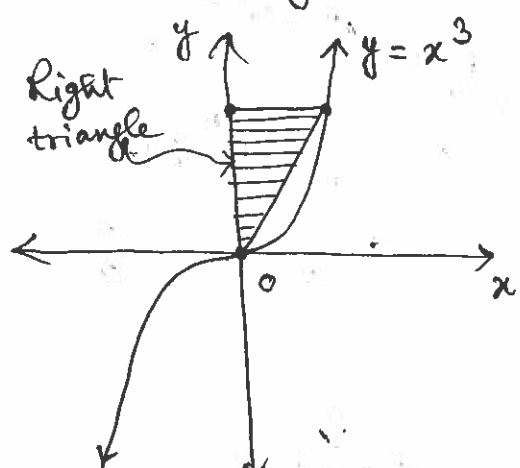
3.6 Mathematical models :

Problems :

5. A right triangle has one vertex on the graph of $y = x^3$, $x > 0$, at (x, y), another at the origin, and the third on the positive y-axis at $(0, y)$. Express the area A of the triangle as a function of x.

Area of a right triangle

$$A = \frac{1}{2}xy = \frac{1}{2}x \cdot x^3 = \frac{x^4}{2}$$



10. A circle of radius r is inscribed in a square.

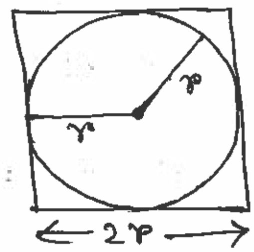
(a) Express the area A of the square as a function of the radius of the circle.

(b) Express the perimeter p of the square as a function of r.

(a) length of the sides of the square $= 2r$ unit.

The area of the square $= 2r \cdot 2r$ unit^2

$= 4r^2$ unit^2.



(b) Perimeter of the square $= 4(2r)$ unit

$= 8r$ unit.

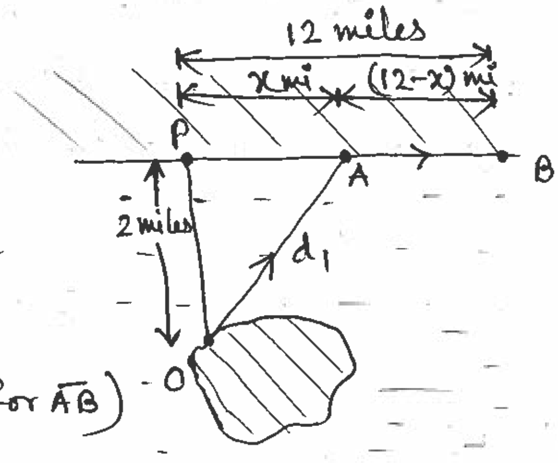
23. An island is 2 miles from the nearest point P on a straight shoreline. A town is 12 miles down the shore from P.

(a) If a person can row a boat at an average speed of 3 miles per hour and the same person can walk 5 miles per hour, build a model that expresses the time T that it takes to go from the island to town as a function of the distance x from P to where the person lands the boat.

(b) What is the domain of T?

(c) How long will it take to travel from the island to town if the person lands the boat 4 miles from P?

(d) How long will it take if the person lands the boat 8 miles from P?



(a) let us consider that person takes t_1 hour to reach to the shoreline A from O and walks along the shoreline from A to B in t_2 hour.

Total time $T = t_1$ (for $\overline{OA}$) $+ t_2$ (for $\overline{AB}$)

Consider the distance $\overline{OA} = d_1$ miles.

As the speed of the boat is 3 miles/hour, and it takes t_1 hour to reach A, $\quad d_1 = 3t_1$ miles.

$\triangle OPA$ is a right triangle, therefore applying Pythagorean theorem —

$$\overline{OA}^2 = \overline{OP}^2 + \overline{PA}^2$$

$$\Rightarrow d_1^2 = 2^2 + x^2$$

$$\Rightarrow (3t_1)^2 = 2^2 + x^2 \qquad (\text{As } d_1 = 3t_1)$$

$$\Rightarrow t_1 = \sqrt{\frac{2^2+x^2}{9}}$$

$$\therefore t_1 = \frac{1}{3}\sqrt{4+x^2} \text{ hour}$$

As, the person can walk ~~from~~ at a speed of 5 miles/hour from A to B and he takes t_2 hour to reach at B,

$$\overline{AB} = 5t_2 \text{ miles.}$$

$$\therefore 5t_2 = 12 - x$$

$$\Rightarrow t_2 = \frac{12-x}{5}$$

$\therefore$ Total time taken to reach the town B from O via a is

$$T = t_1 + t_2 = \frac{1}{3}\sqrt{4+x^2} + \frac{12-x}{5} \text{ hour.}$$

(b) As the time T is defined for any real value of x, the domain of T is $\rightarrow$ $x \in \mathbb{R}$ (any real number).

(c) ~~If the speed of the boat would be~~,
If the person lands the boat 4 miles from P, then

$$x = 4$$

$$\therefore T = t_1 + t_2 = \frac{1}{3}\sqrt{4+x^2} + \frac{12-x}{5} = \frac{1}{3}\sqrt{4+4^2} + \frac{12-4}{5}$$

$$= \frac{1}{3}2\sqrt{5} + \frac{8}{5} \text{ hour.}$$

(d) If $x = 8$

$$T = \frac{1}{3}\sqrt{4+x^2} + \frac{12-x}{5} = \frac{1}{3}\sqrt{4+8^2} + \frac{12-8}{5} = \frac{1}{3}2\sqrt{17} + \frac{4}{5} \text{ hour.}$$