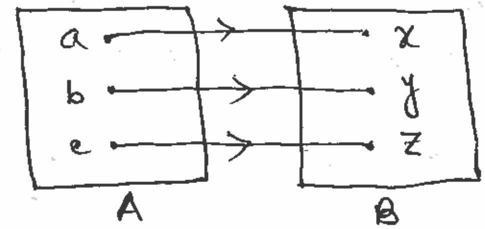
3.1 Functions:

Relation: A relation is a correspondence between two sets. If a, b, c are the elements of set A and x, y, z are elements of set B; and there are some correspondence between the elements of the two sets, then it is said that there is a relation between two sets.

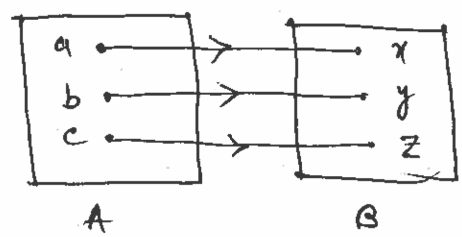
For example: if 'a' corresponds to 'x' then one writes $a \to x$.

A relation can be expressed through a technique called mapping.

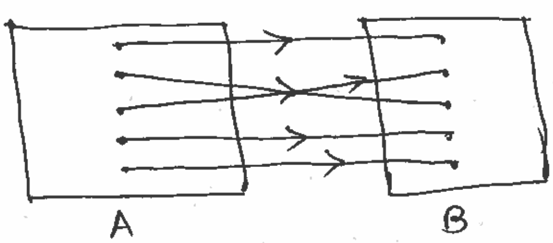


Mapping between A and B

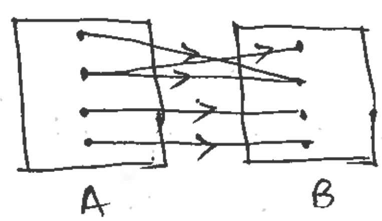
Mapping: A map is a way of associating unique objects to every element in a given set. A map $f: A \mapsto B$ from A to B is such that for every $a, b, c \in$ (Belongs to) A, there is a unique object $x, y, z \in B$.



A mapping can be of different types, where every elements of one set may correspond to every elements of the other set. If there is a one-to-one relationship between the elements of the two sets, then the mapping is called injective mapping. If ~~there~~ every element of set A corresponds to one or more than one elements of set B, then the mapping is called surjective mapping.



Injective mapping.



Surjective mapping.

Function: If there is a one-to-one relationship between two sets A and B, which means set A maps to set B injectively, then set B is called the function of set A. If one element of A corresponds to more than one elements of set B, then B is not a function of A.

Example:

$$y = x + 10$$

For every x, there is a unique value of y. for example for $x = 1$, $y = 11$, $x = -10$, $y = 0$ etc. A single value of x gives a single value of y, that is why one can say that y is a function of x and is denoted by

$$y = f(x) \rightarrow y \text{ equals function of } x.$$

- $x^2 + y^2 = 1$

$$\Rightarrow y^2 = -x^2 + 1$$

$$\Rightarrow y = \pm\sqrt{-x^2+1}$$

$$= \pm\sqrt{1-x^2}$$

Domain of x: $|(1-x^2)| \geq 0$

$$\Rightarrow \pm(1-x^2) \geq 0$$

$$(1-x^2) \geq 0 \qquad -(1-x^2) \geq 0$$

$$\Rightarrow x^2 \leq 1 \qquad \Rightarrow x^2 \geq 1$$

$$\therefore -1 \leq x^2 \leq 1.$$

Domain:

$$|x^2| \geq 1$$

$$\Rightarrow \pm x^2 \geq 1$$

$$x^2 \geq 1 \qquad -x^2 \geq 1$$

$$\Rightarrow x^2 \leq -1$$

$$\therefore -1 \leq x^2 \leq 1.$$

Consider a value of x^2 inside the domain, say $x^2 = 0$, then $y = \pm 1$. That is the reason that y is not a function of x, because for every value of x, there are more than one value of y.

In a function $y = f(x)$, x is called independent variable, whereas y is called a dependent variable. The reason is the variable y depends on the values of x.

If one says that y is a function

$$y = f(x) = x + 10 \text{, then it means}$$

$$y = f(2) = 2 + 10 = 12$$
$$y = f(3) = 3 + 10 = 13$$
$$y = f\left(\tfrac{3}{2}\right) = \tfrac{3}{2} + 10 = \tfrac{23}{2} \quad \text{etc.}$$

Properties of function :

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
4. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$; $g(x) \neq 0$

Problems :

28. Determine whether the equation defines y as a function of x. $y = x^3$

This is clear, that for a single value of x there exists a single value of y. for example

if $x = 2$, $y = 8$

if $x = -2$, $y = -8$

Therefor, y can be defind as a function of x.

31. $y^2 = 4 - x^2$

$\Rightarrow y = \pm\sqrt{4 - x^2}$

For every x, there are two values of y. Therefore y is not a funcion of x in this case.

39. $f(x) = 3x^2 + 2x - 4$, find $f(0)$, $f(1)$, $f(-1)$, $f(-x)$, $-f(x)$, $f(x+1)$, $f(2x)$, $f(x+h)$

$$f(x) = 3x^2 + 2x - 4$$

① $\therefore f(0) = 3 \cdot 0^2 + 2 \cdot 0 - 4 = -4$

② $f(1) = 3 \cdot 1^2 + 2 \cdot 1 - 4 = 1$

③ $f(-1) = 3 \cdot (-1)^2 + 2 \cdot (-1) - 4 = -3$

④ $f(-x) = 3(-x)^2 + 2(-x) - 4 = 3x^2 - 2x - 4$

⑤ $-f(x) = -(3x^2 + 2x - 4) = -3x^2 - 2x + 4$

⑥ $f(x+1) = 3 \cdot (x+1)^2 + 2(x+1) - 4$

$$= 3x^2 + 6x + 3 + 2x + 2 - 4$$

$$= 3x^2 + 8x + 1$$

⑦ $f(2x) = 3 \cdot (2x)^2 + 2 \cdot 2x - 4$

$$= 12x^2 + 4x - 4$$

⑧ $f(x+h) = 3 \cdot (x+h)^2 + 2(x+h) - 4$

$$= 3x^2 + 6xh + h^2 + 2x + 2h - 4$$

$$= 3x^2 + h^2 + 3xh + 2x + 2h - 4.$$

48. Find the domain of the function $f(x) = x^2 + 2$

Since the operation can be performed on any real numbers, we say that the domain of the function is any real number. i.e. $f(x) \in \mathbb{R}$

59. Find the domain of the function $P(x) = \sqrt{\frac{2}{x-1}}$

$$P(x) = \sqrt{\frac{2}{x-1}} = \frac{\sqrt{2}}{\sqrt{x-1}}$$

① $\sqrt{x-1}$ has to be greater or equals zero. $\therefore (x-1) \geq 0$

$$\Rightarrow x \geq 1$$

② $\frac{\sqrt{2}}{\sqrt{x-1}}$ is not defined when the denominator is zero, therefore, $x = 1$ is not allowed. So the domain is $x > 1$

74. Given $f(x) = \frac{1}{x}$ and $\left(\frac{f}{g}\right)(x) = \frac{x+1}{x^2-x}$. Find the function g.

$$\left(\frac{f}{g}\right)(x) = \frac{x+1}{x^2-x}$$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{x+1}{x^2-x}$$

$$\Rightarrow \frac{1/x}{g(x)} = \frac{x+1}{x^2-x}$$

$$\Rightarrow g(x)\cdot(x+1) = \frac{1}{x}\cdot(x^2-x)$$

$$\Rightarrow g(x)\cdot(x+1) = (x-1)$$

$$\Rightarrow g(x) = \frac{x-1}{x+1}$$

87. If $f(x) = \frac{2x - \text{[redacted]}A}{x - \text{[redacted]}3}$ and $f(4) = 0$, what is the value of A? where is f not defined?

$$f(x) = \frac{2x - \text{[redacted]}A}{x - \text{[redacted]}3}$$

$$\Rightarrow f(4) = \frac{2\cdot 4 - \text{[redacted]}A}{4 - \text{[redacted]}3}$$

$$\Rightarrow 0 = \frac{8 - \text{[redacted]}A}{4 - \text{[redacted]}3} \quad (\text{as } f(4) = 0)$$

$$\Rightarrow 0 = \frac{8 - A}{1}$$

$$\Rightarrow 8 - A = 0$$

$$\Rightarrow A = 8$$

[redacted] 91. Express the gross salary G of a person, who earns \$10 per hour as a function of the number x of hours worked.

Gross salary = \$10 for one hour.

Gross salary = \$$10 \cdot x$ for x hours.

$\therefore$ $G(x) = 10x$.

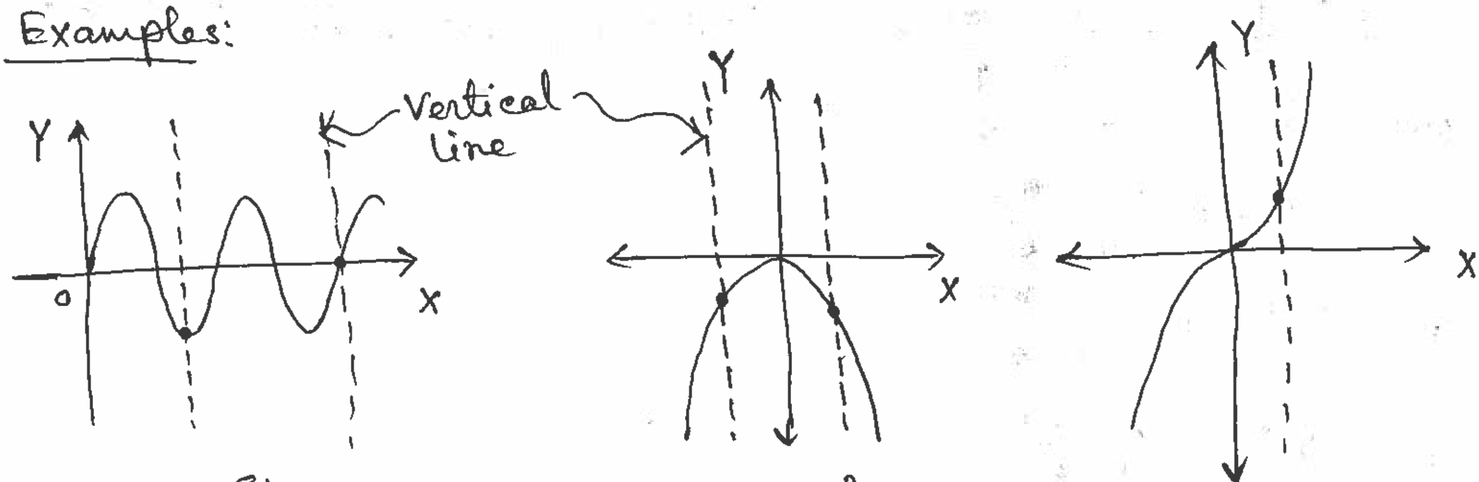
3.2 The graph of a function:

Not every collection of points in the x-y plane represents the graph of a function. Remember, for a function, each number x in the domain has exactly one image y in the range. This means that the graph of a function cannot contain two points with the same x-coordinate.

Vertical-line test:

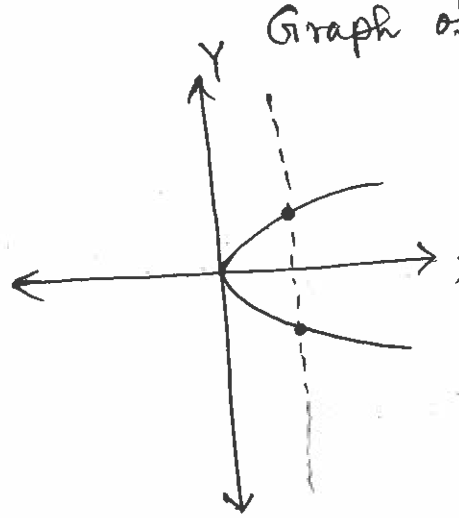
Draw a vertical line on the graph and see if that intersects the graph at one point or more than one points. If the vertical line intersects the graph at more than one points, then the graph is not a ~~fun~~ graph of function, otherwise it is.

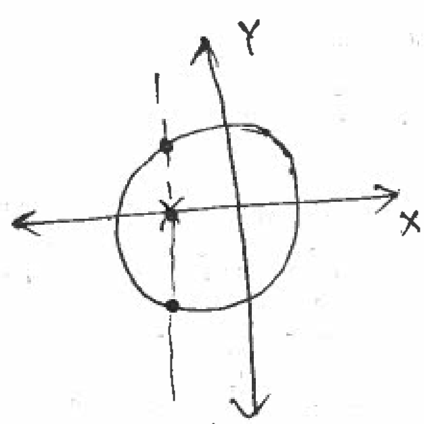
Examples:



$y = \sin x$ $\qquad$ $y = -x^2$ $\qquad$ $y = x^3$

Graph of a function.





$x^2 + y^2 = 1$

$y^2 = x$ $\qquad$ Not graph of a function.

Problems:

24. $f(x) = -3x^2 + 5x$

(a) Is the point $(-1, 2)$ on the graph of $f(x)$?
(b) If $x = -2$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = -2$, what is x? What points are on the graph f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f?
(f) List the y-intercepts, if there is one, of the graph of f!

(a) $f(x) = -3x^2 + 5x$

When $x = -1$, then
$$f(x) = -3x^2 + 5x$$
$$f(-1) = -3(-1)^2 + 5(-1)$$
$$= -3 - 5$$
$$= -8$$

$\therefore$ The point $(-1, -8)$ is on the graph but the point $(-1, 2)$ is not.

(b) If $x = -2$, then
$$f(-2) = -3(-2)^2 + 5(-2)$$
$$= -12 - 10$$
$$= -22$$

$\therefore$ The point $(-2, -22)$ is on the graph.

(c) If $f(x) = -2$, then
$$-3x^2 + 5x = -2$$
$$\Rightarrow 3x^2 - 5x - 2 = 0$$
$$\Rightarrow 3x^2 - 6x + x - 2 = 0$$
$$\Rightarrow 3x(x-2) + 1(x-2) = 0$$
$$\Rightarrow (x-2)(3x+1) = 0$$

$\therefore$ Either, $x - 2 = 0$ or, $3x + 1 = 0$

$\Rightarrow x = 2$ $\qquad \Rightarrow x = -\frac{1}{3}$

$\therefore$ The points, which are on the graph $(2, -2)$ and $(-\frac{1}{3}, -2)$

(d) Domain of of f is the real number, because the function is defined for any real number.

(e) To find x-intercepts, one needs to put $y = f(x) = 0$

$\therefore -3x^2 + 5x = 0$

$\Rightarrow x(-3x+5) = 0$

$\therefore$ Either, $x = 0$

or, $-3x+5 = 0$

$\Rightarrow x = \frac{5}{3}$

$\therefore$ x-intercepts are $(0,0)$ and $(\frac{5}{3}, 0)$

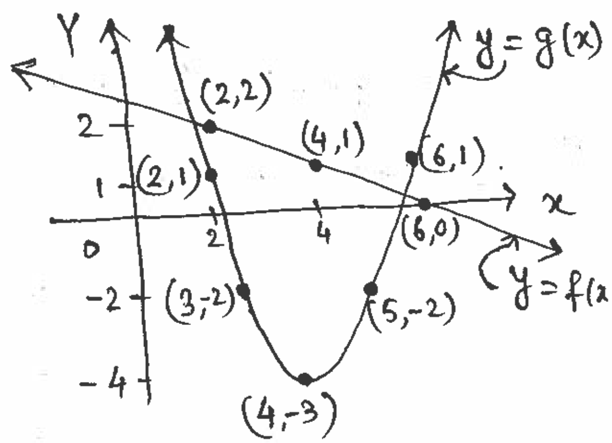
(f) To find y-intercepts, put $x = 0$

$f(x) = -3\cdot 0^2 + 5\cdot 0$

$= 0$

$\therefore$ The y-intercept is only at $(0,0)$. Of course, you can not have more than one y-intercept, because in that case the graph will not be a graph of function.

<u>35</u>. The graph of two functions, f and g is illustrated. Use the graph to answer (a) $(f+g)(2)$ (b) $(f+g)(4)$ (c) $(f-g)(6)$ (d) $(g-f)(6)$ (e) $(f\cdot g)(2)$ (f) $(\frac{f}{g})(4)$



(a) $(f+g)(2) = f(2) + g(2) = 2 + 1 = 3$

(b) $(f+g)(4) = f(4) + g(4) = 1 - 3 = -2$

(c) $(f-g)(6) = f(6) - g(6) = 0 - 1 = -1$

(d) $(g-f)(6) = g(6) - f(6) = -1 - 0 = -1$

(e) $(f\cdot g)(2) = f(2)\cdot g(2) = 2\cdot 1 = 2$

(f) $(\frac{f}{g})(4) = \frac{f(4)}{g(4)} = \frac{1}{-3} = -\frac{1}{3}$.