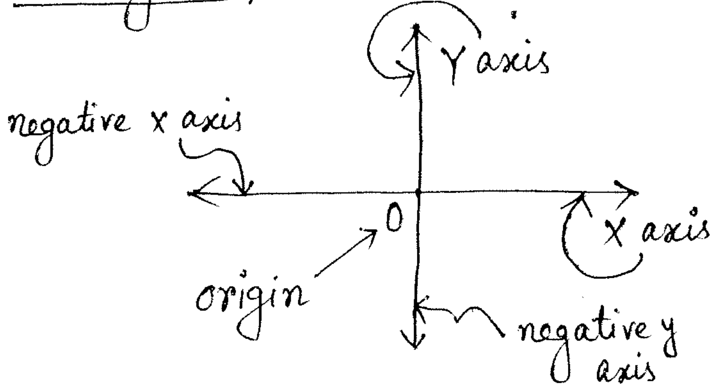
Lecture-4 The distance and midpoint formulas 28.01.15

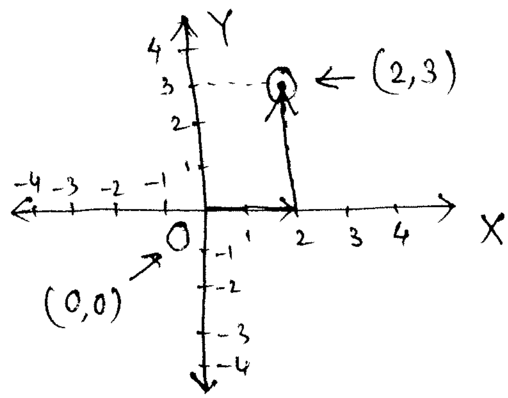
Rectangular / Cartesian coordinate system :



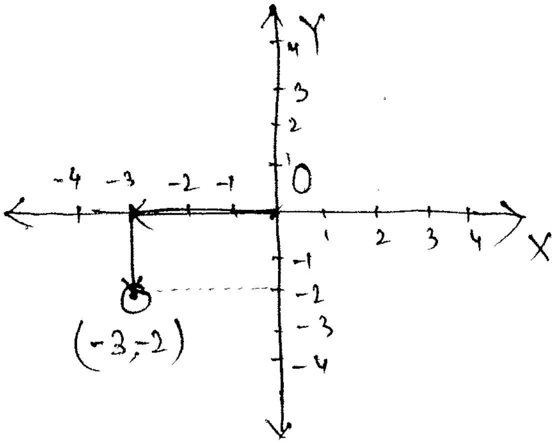
X axis and Y axis are situated in the same plane, a two dimensional plane, called X-Y plane.

A point in this plane denoted by (_,_), where the vacent positions are filled up by two numbers, is called co-ordinate of the system. The coordinate tells the position of the point in the plane.

For example: Consider a coordinate (2, 3), which means x-coordinate or abscissa is '2' and the y-coordinate or ordinate is '3'. Let us now see the position of the co-ordinate in the Rectangular or cartisian coordinate system. The coordinate of the origin is always (0,0)

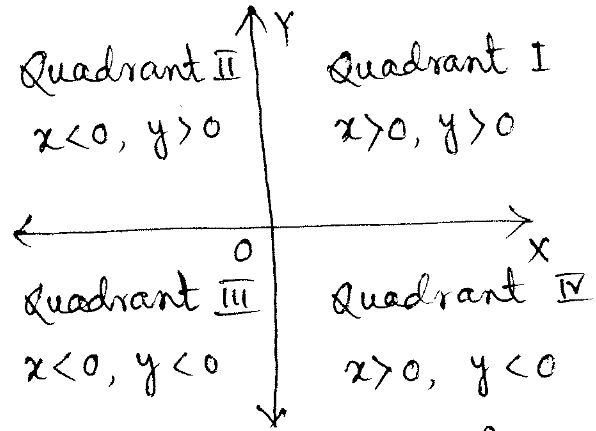


Similarly one can plot the coordinate (−3, −2) in the same coordinate system as shown below −

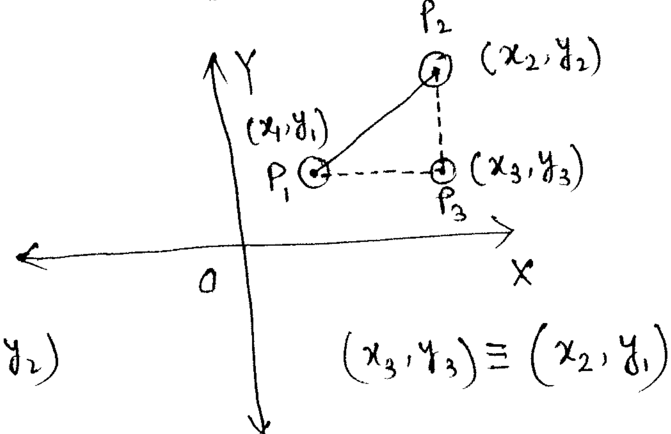


$(-3,-2) \equiv (x, y)$, means one needs to travel the x-coordinate first by (−3) unit then (−2) unit towards the y-coordinate.

The XY plane in the cartesian co-ordinate system is divided into four sections by the X and Y axis. These four sections are called <u>quadrants</u>.



<u>Distance between two points :</u>



Given $\begin{cases} P_1 \to (x_1, y_1) \\ P_2 \to (x_2, y_2) \end{cases}$

One needs to find the distance between the two points P_1 and P_2, whose coordinates are (x_1, y_1) and (x_2, y_2) respectively.

We consider a point P_3 on the same coordinate system, whose coordinates can be found out by looking at the picture. clearly $P_3 \equiv (x_3, y_3) \equiv (x_2, y_1)$.

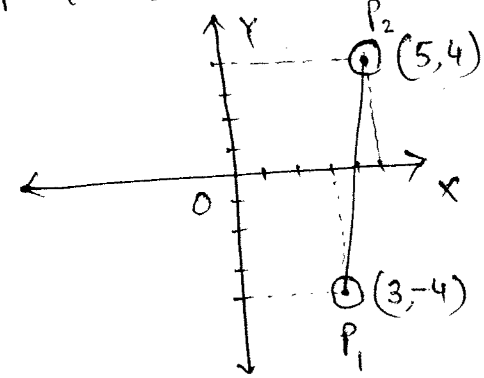
The horizontal distance between P_1 and P_3 is therefore $d(P_1, P_3) = |x_2 - x_1|$ and the vertical distance between P_3 and P_2 is $d(P_3, P_2) = |y_2 - y_1|$. Therefore using Pythagorean Theorem we can easily obtain the distance between two points P_1 and P_2 as —

$$[d(P_1, P_2)]^2 = [d(P_3, P_2)]^2 + [d(P_1, P_3)]^2$$
$$= |y_2 - y_1|^2 + |x_2 - x_1|^2$$
$$= (y_2 - y_1)^2 + (x_2 - x_1)^2$$
$$\therefore d(P_1, P_2) = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Pythagorean Theorem
$(\text{Hypotenuse})^2 = (\text{Base})^2 + (\text{height})^2$, as the triangle $\Delta P_1 P_2 P_3$ is a right angled triangle.

Problems:

21. Find the distance between two points $P_1 = (3,-4)$; $P_2 = (5,4)$



Formula : $d(P_1,P_2) = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$

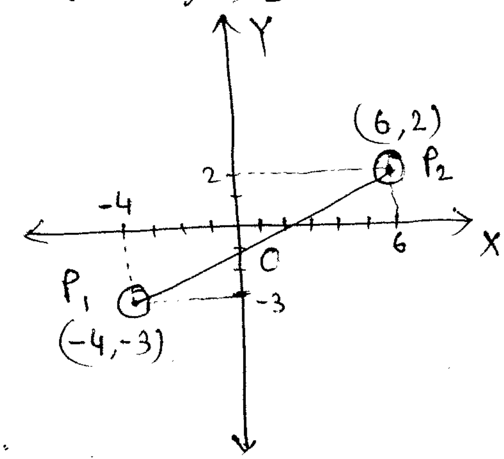
$\left.\begin{matrix} x_1 = 3 \\ y_1 = -4 \end{matrix}\right\}$ $\quad \left.\begin{matrix} x_2 = 5 \\ y_2 = 4 \end{matrix}\right\}$

$\therefore \; d(P_1,P_2) = \sqrt{\{4-(-4)\}^2 + \{5-3\}^2}$

$= \sqrt{8^2 + 2^2} = \sqrt{68} = \sqrt{2 \times 2 \times 17} = 2\sqrt{17}$ unit.

26. Find the distance between two points $P_1 = (-4,-3)$; $P_2 = (6,2)$

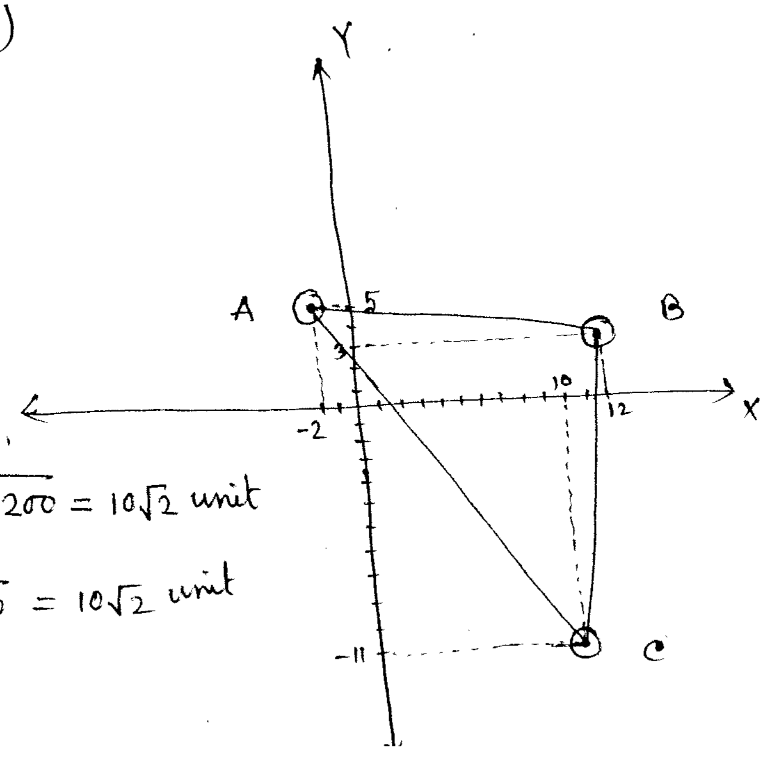
$\left.\begin{matrix} x_1 = -4 \\ y_1 = -3 \end{matrix}\right\}$ $\quad \left.\begin{matrix} x_2 = 6 \\ y_2 = 2 \end{matrix}\right\}$



$\therefore \; d(P_1 P_2) = \sqrt{\{2-(-3)\}^2 + \{6-(-4)\}^2}$

$= \sqrt{5^2 + (10)^2} = \sqrt{125} = 5\sqrt{5}$ unit.

30. Plot each point and form the triangle ABC.
Verify that the triangle is a right triangle. Find its area.
$A = (-2,5)$; $B = (12,3)$; $C = (10,-11)$



$d(A,C) = \sqrt{\{(10)-(-2)\}^2 + \{(-11)-5\}^2}$

$= \sqrt{12^2 + (-16)^2} = \sqrt{144 + 256}$

$= \sqrt{400} = 20$ unit

$d(A,B) = \sqrt{14^2 + (-2)^2} = \sqrt{196+4} = \sqrt{200} = 10\sqrt{2}$ unit

$d(B,C) = \sqrt{(-2)^2 + (-14)^2} = \sqrt{4+196} = \sqrt{200} = 10\sqrt{2}$ unit

$$[d(A,C)]^2 = 20 \times 20 = 400 \text{ unit}^2$$

$$[d(A,B)]^2 = 10\sqrt{2} \times 10\sqrt{2} = 200 \text{ unit}^2$$

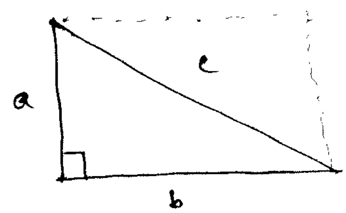
$$[d(B,C)]^2 = 10\sqrt{2} \times 10\sqrt{2} = 200 \text{ unit}^2$$

As, $[d(A,C)]^2 = [d(A,B)]^2 + [d(B,C)]^2$, the triangle is a right-angled triangle.

~~Finding midpoint:~~

<u>Area</u>: Area of a right angled triangle

$$= \frac{ab}{2} \text{ unit}^2$$



∴ The area of the triangle

$$= \frac{d(A,B)\, d(B,C)}{2} = \frac{\sqrt{200 \times 200}}{2} = 100 \text{ unit}^2.$$

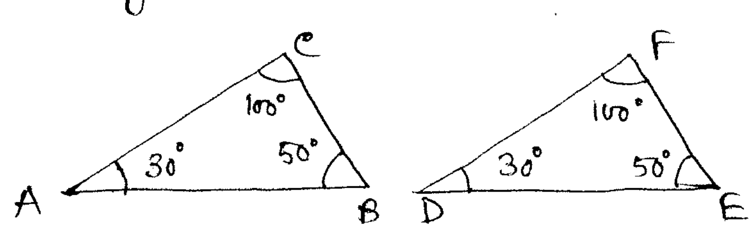
<u>Finding midpoint</u>:

Before finding the formula for the midpoint of a line, let us look at some basic geometry.

1. <u>Congruent triangles</u>:

Two triangles are congruent if each of the corresponding angles is the same measure and each of the corresponding sides is the same length.

Two triangles are congruent because



$\angle CAB = \angle FDE$

$\angle ABC = \angle DEF$

$\angle ACB = \angle DFE$

and $\overline{AB} = \overline{DE}$, $\overline{AC} = \overline{DF}$ and $\overline{BC} = \overline{EF}$.

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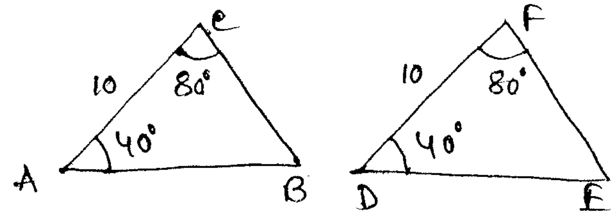
(i) Angle - side - angle case :



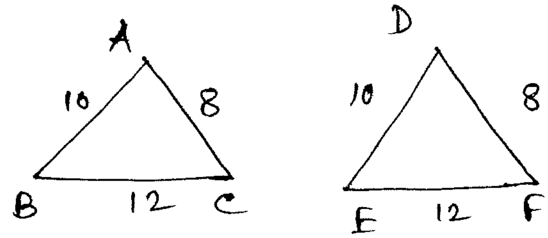
If at least

$\angle CAB = \angle FDE$

$\angle ACB = \angle DFE$

and $\overline{AC} = \overline{DF}$, then the two triangles $\triangle ABC$ and $\triangle DEF$ are said to be congruent.

(ii) side - side - side case :



If,

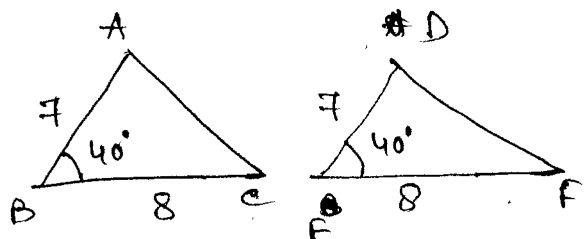
$\overline{AB} = \overline{DE}$

$\overline{BC} = \overline{EF}$

$\overline{AC} = \overline{DF}$

then two triangles $\triangle ABC$ and $\triangle DEF$ are congruent.

(iii) side - angle - side case :



If $\overline{AB} = \overline{DE}$

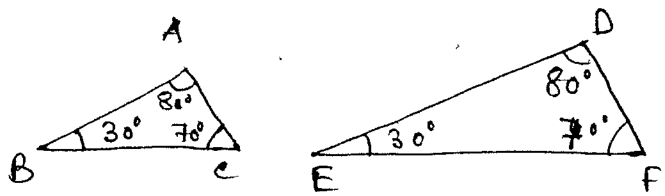
$\angle ABC = \angle DEF$

$\overrightarrow{BC} = \overline{EF}$

then triangles $\triangle ABC$ and $\triangle DEF$ are congruent.

2. Similar triangles :

Two triangles are similar if the corresponding angles are equal and the lengths of the corresponding sides are proportional.



If

$\angle ABC = \angle DEF$

$\angle BCA = \angle EFD$

$\angle CAB = \angle FDE$

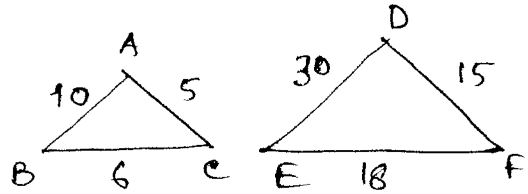
and $\dfrac{\overline{AB}}{\overline{DE}} = \dfrac{\overline{BC}}{\overline{EF}} = \dfrac{\overline{AC}}{\overline{DF}} = \text{Constant}$.

(i) <u>Angle-angle case</u>:

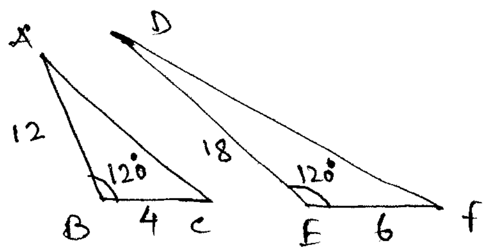
If at least two of the corresponding angles are equal.

(ii) <u>side-side-side case</u>:

$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}} = \frac{1}{3}$$



(iii) <u>side-angle-side case</u>:



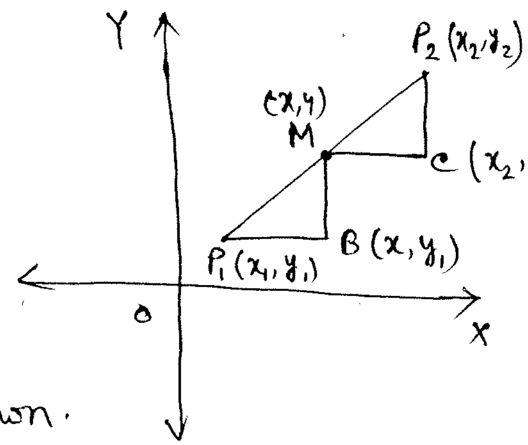
$$\frac{\overline{AB}}{\overline{DE}} = \frac{12}{18} = \frac{2}{3}$$

$$\frac{\overline{BC}}{\overline{EF}} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore \frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{2}{3}$$

and $\angle ABC = \angle DEF = 120°$.

<u>Midpoint formula</u>:



Let us consider a line $\overline{P_1P_2}$, whose midpoint is M. We want to calculate the formula of obtaining the co-ordinates of the midpoint M, if the coordinates of the end points of the line $\overline{P_1P_2}$ are known.

i.e. $\left.\begin{array}{l} P_1 = (x_1, y_1) \\ P_2 = (x_2, y_2) \end{array}\right\}$ known. $\Rightarrow$ calculate $M = (x, y)$

Now, (i) $\overline{P_1M} = \overline{MP_2}$ (Given)

(ii) $\angle MP_1B = \angle P_2MC$

(iii) $\angle P_1MB = \angle MP_2C$

$\therefore$ $\triangle P_1BM$ and $\triangle MCP_2$ are congruent according to the rule - angle-side-angle.

As, the triangle are congruent, corresponding sides are equal in length.

i.e. $\overline{P_1 A} = \overline{MC}$ and $\overline{MB} = \overline{P_2 C}$

$\Rightarrow x - x_1 = x_2 - x$ $\qquad \Rightarrow y - y_1 = y_2 - y$

$\Rightarrow x + x = x_2 + x_1$ $\qquad \Rightarrow 2y = y_1 + y_2$

$\Rightarrow 2x = x_1 + x_2$ $\qquad \Rightarrow y = \frac{y_1 + y_2}{2}$

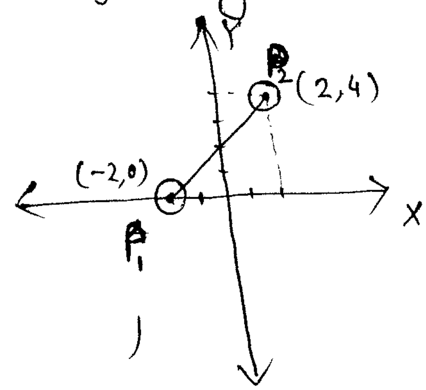
$\Rightarrow x = \frac{x_1 + x_2}{2}$

$\therefore$ The coordinate of the midpoint is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Problem

36. Find the midpoint of the line segment joining the points P_1 and P_2, $P_1 = (-2, 0)$; $P_2 = (2, 4)$



The midpoint is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$\begin{cases} x_1 = -2 \\ y_1 = 0 \end{cases}$ $\qquad \begin{cases} x_2 = 2 \\ y_2 = 4 \end{cases}$

$$\therefore M = (x, y) = \left(\frac{-2+2}{2}, \frac{0+4}{2} \right) = (0, 2)$$

2.2 Graphs of equations in two variables; intercepts; symmetry

Graph equations by plotting points:

We consider an equation

$$2x - y = 6$$

We want to plot the equation in a graph paper.

$2x - y = 6$

$\Rightarrow 2x = 6 + y$

$\Rightarrow x = \dfrac{6+y}{2}$

Now, we choose various values of y and calculate corresponding values of x from the above equation. Let us make a table.

x	3	4	5	6	2	1	0
y	0	2	4	6	-2	-4	-6

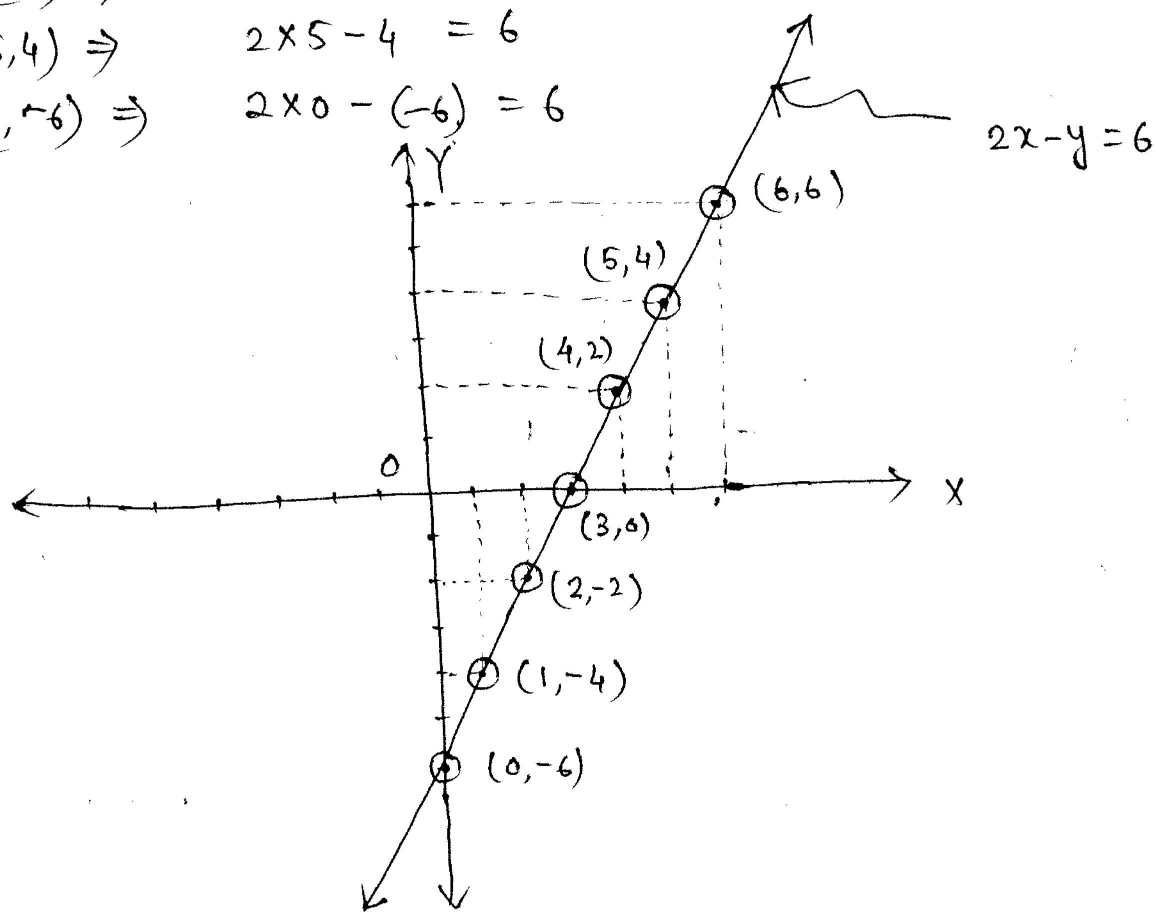
That means all 7 pair of points are solutions of the equation $2x - y = 6$. Let us first verify few of them.

Consider $(3,0) \Rightarrow 2 \times 3 - 0 = 6$

$(4,2) \Rightarrow 2 \times 4 - 2 = 6$

$(5,4) \Rightarrow 2 \times 5 - 4 = 6$

$(0,-6) \Rightarrow 2 \times 0 - (-6) = 6$



Determine whether a point is on the graph of an equation

Problem 15. Equation : $x^2+y^2=4$, determine which of the following points are on the graph of the equation.

Points : $(0,2)$; $(-2,2)$; $(\sqrt{2},\sqrt{2})$

First we put the first point $(0,2)$ in the equation and see whether the equation is satisfied by the point

$$x^2+y^2 = (0)^2+(2)^2 = 4 \leftarrow \text{satisfied.}$$

2nd point : $(-2,2) \Rightarrow x^2+y^2 = (-2)^2+(2)^2 = 4+4=8 \neq 4$
Not satisfied.

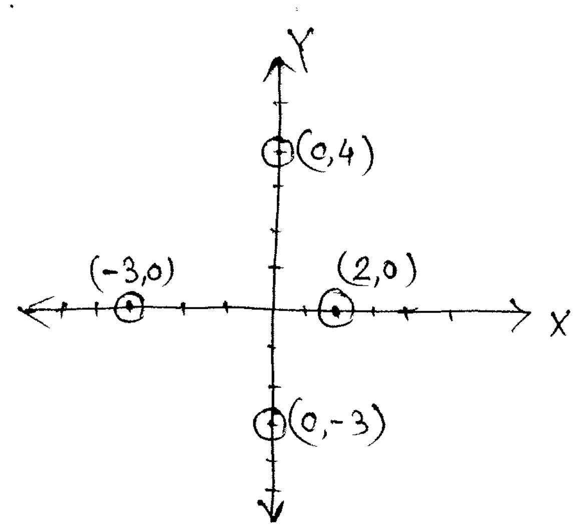
3rd point : $(\sqrt{2},\sqrt{2}) \Rightarrow (\sqrt{2})^2+(\sqrt{2})^2 = 2+2=4 \leftarrow$ satisfied.

$\therefore$ The points, $(0,2)$ and $(\sqrt{2},\sqrt{2})$ are on the graph $x^2+y^2=4$, but the point $(-2,2)$ is not on the graph.

Find intercepts from a graph :

One needs to find the points from the given graph which intersects either x-axis or y-axis or both. One should note that the equation of x-axis is $y=0$ and the equation of y-axis is $x=0$. Because on x axis the value of y-coordinate is zero and vice-versa.

Do you see why? Have a look at the graph.



Finding intercepts from an equation:

1. To find x-intercepts of an equation, one needs to set $y=0$ first and then solve the equation for x.

2. To find intercepts of an equation, one needs to set $x=0$ first and then solve the equation for y.

Problems:

18. Find the intercepts and graph each equation by plotting the points. $y = x-6$

x-intercept:

$$y = x-6$$
$$\Rightarrow 0 = x-6 \quad \leftarrow \text{Set } y=0$$
$$\Rightarrow -x = -6$$
$$\Rightarrow x = 6$$

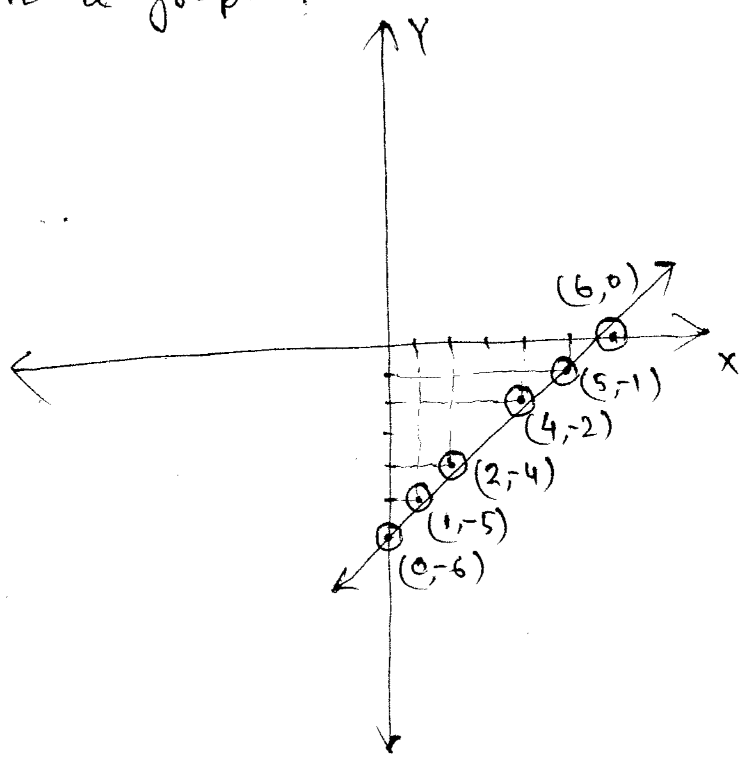
y-intercept:

$$y = x-6$$
$$\Rightarrow y = 0-6 \quad \leftarrow \text{Set } x=0$$
$$\Rightarrow y = -6$$

Let us plot the equation now on a graph:

$$y = x-6$$

x	0	1	2	6	4	5
y	-6	-5	-4	0	-2	-1



Now we can see the intercepts from the graph also

x-intercept is at $x=6$

y-intercept is at $y=-6$

Test an equation for symmetry with respect to the x-axis, the y-axis and the origin.

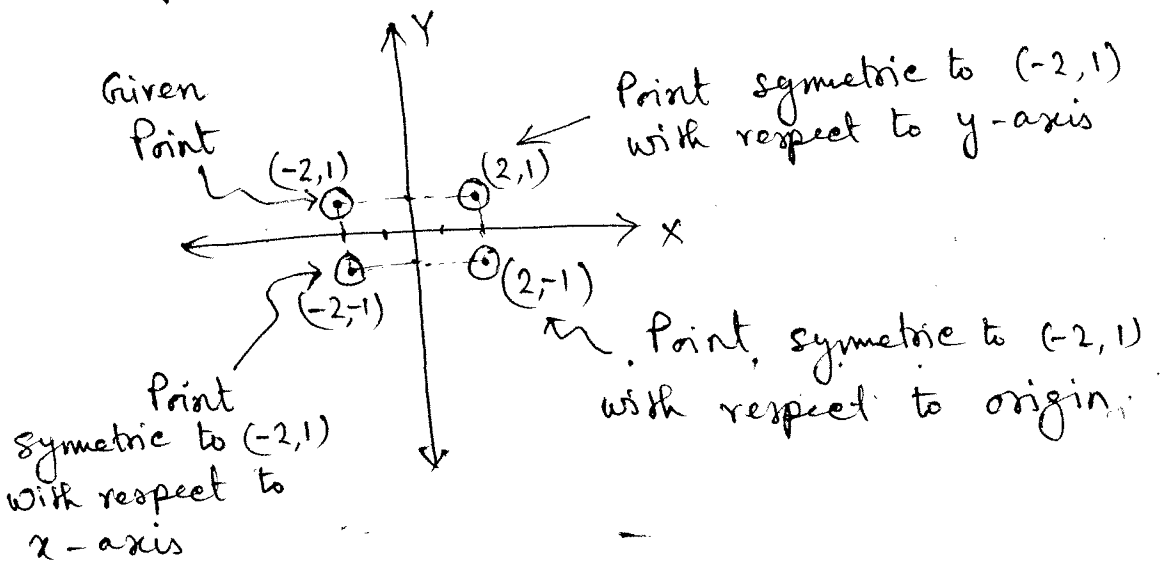
A graph is said to be symmetric with respect to the x axis if, for every point (x, y) on the graph, the point $(x, -y)$ is also on the graph.

Similarly, a graph is said to be symmetric with respect to the y-axis if, for every point (x, y) on the graph, the point $(-x, y)$ is also on the graph.

A graph is said to be symmetric with respect to the origin, if, for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph.

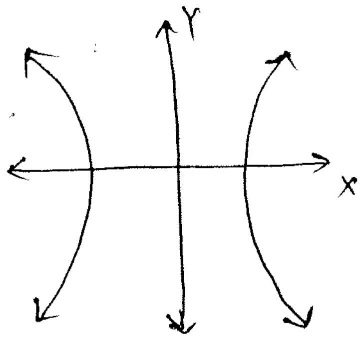
Problems

31. Plot the point $(-2, 1)$. Then plot the point that is symmetric to it with respect to (a) x-axis (b) y-axis (c) the origin.



(*) The easiest way to find whether a point is symmetric to a point or not is - place a mirror across the axis or at origin and check whether you can see a reflection. for example: the point $(2,1)$ is a mirror reflection of the point $(-2,1)$ when one places a mirror across y-axis. That means the point $(2,1)$ is symmetric to point $(-2,1)$ with respect to y-axis.

Probem 39



If one puts a mirror across y axis then one graph is the reflection of the other. Therefore the graph is symmetric to Y-axis.

64. list the intercepts and test for symmetry. $y = x^4 - 1$

x-intercept: $y = x^4 - 1$

$\Rightarrow 0 = x^4 - 1 \leftarrow$ Put $y = 0$

$\Rightarrow x^4 = 1$

$\Rightarrow x = 1$

y-intercept: $y = x^4 - 1$

$\Rightarrow y = 0 - 1 \leftarrow$ Put $x = 0$

$\Rightarrow y = -1$

Symmetry with respect to x-axis:

Replace y by $(-y)$ in the equation $y = x^4 - 1$, then the equation becomes $-y = x^4 - 1$, which is not equivalent to the original equation. Therefore the equation is not symmetric to x axis.

Symmetry with respect to y-axis:

Replace x by $(-x)$ in $y = x^4 - 1$, which gives $y = x^4 - 1$, which is equivalent to original equation. Therefore the equation is symmetric to y axis.

Symmetry with respect to origin:

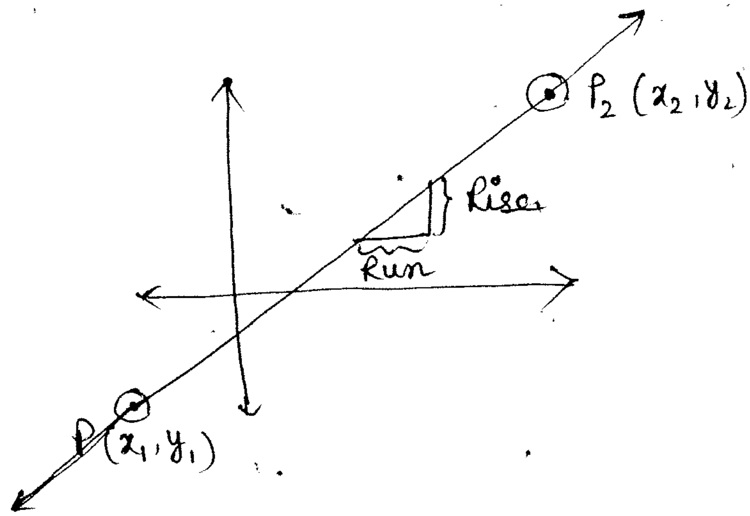
Replace x by $(-x)$ and y by $(-y)$, then see if the equation is equivalent to the original equation.

2·3 Lines

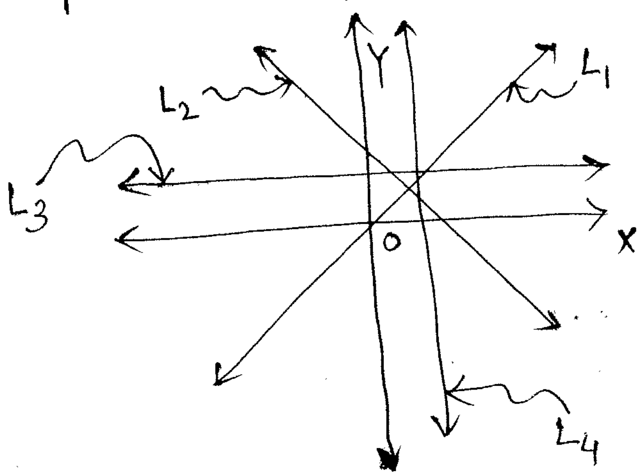
Calculate slope of a line:

Slope m is defined by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2 .$$



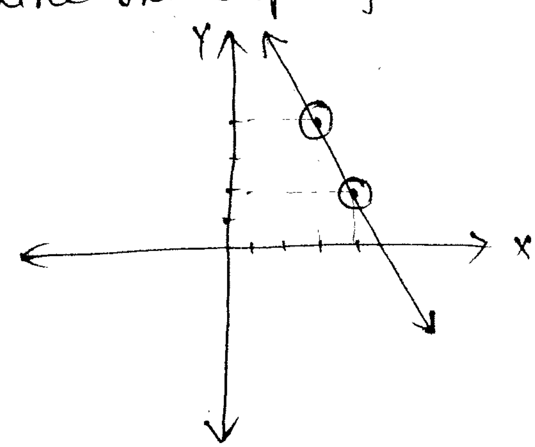
- when the slope of a line is positive, the line slants upward from left to right (L_1).
- when the slope of a line is negative, the line slants downward from left to right (L_2).
- when the slope is 0, the line is horizontal (L_3).
- when the slope is undefined, the line is vertical (L_4).



Problem

16. Plot each pair of point and determine the slope of the line.
(4,2); (3,4) :

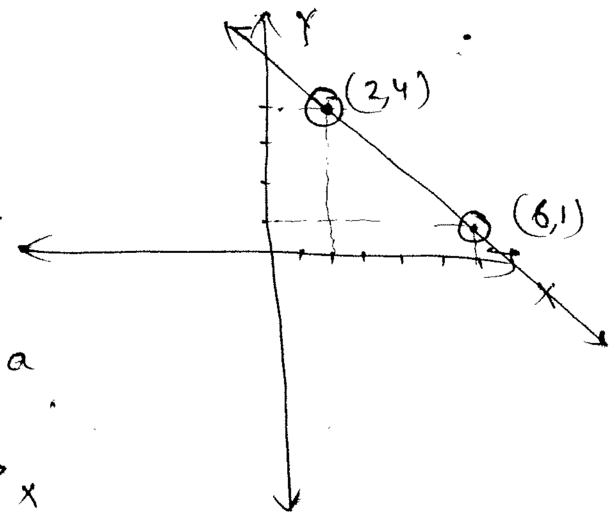
From the figure it is clear that the line has a negative slope.



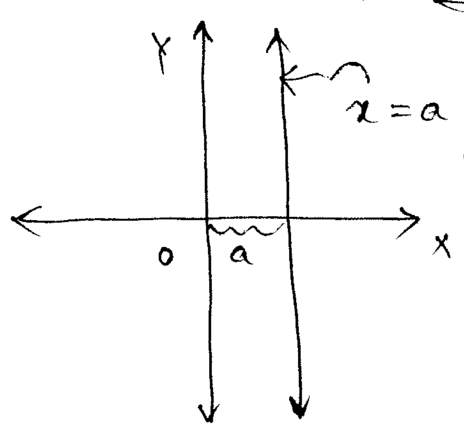
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3-4} = \frac{+2}{-1} = -2$$

25. Graph the line containing the point $P = (2,4)$ and having a slope $m = \frac{-3}{4}$.

Slope $m = \frac{-3}{4} = \frac{\text{Rise}}{\text{Run}}$. That means that for every horizontal movement (run) of 4 units to the right there will be a vertical movement (rise) of (-3) units. If we start at the given point $(2,4)$ and move 4 units to the right and 3 units down, we reach the point $(6,1)$. By drawing the line through these points, we have the graph.



⊛ The equation of a vertical line is $x = a$, where a is the x-intercept.



Equation of a line given a point and slope:

Consider a point $P_1(x_1, y_1)$ is given and the value of the slope (m) is given. Consider another point $P(x,y)$, which is not known. Then

$$m = \frac{y - y_1}{x - x_1}$$

$\Rightarrow y - y_1 = m(x - x_1)$ ← This is the equation of the line.

Finding an equation of a line given two points

Consider the two points $(5,4)$ and $(-2,5)$, then the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-4}{-2-5} = \frac{1}{-7}$$

Then, the equation of line containing a point (5,4) having slope $-\frac{1}{7}$ is

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 4 = -\frac{1}{7}(x-5)$$
$$\Rightarrow 7y - 28 = -x + 5$$
$$\Rightarrow 7y + x = 33$$

Write the equation of a line in slope-intercept form

Consider the slope and y-intercept b are known. That means we know both the slope m of the line and a point (0, b) on the line. Then

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - b = m(x - 0)$$
$$\Rightarrow \boxed{y = \underset{\text{Slope}}{m}x + \underset{\text{Intercept}}{b}} \leftarrow \text{Slope-intercept form of an equation.}$$

Problem

50. Find an equation for the line with the given properties. Express your answer using either the general form or the slope-intercept form of the equation of a line. Points (−3, 4) and (2, 5)

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-4}{2-(-3)} = \frac{1}{5}$$

∴ The equation of the line

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 4 = \frac{1}{5}(x - (-3))$$
$$\Rightarrow y - 4 = \frac{1}{5}(x+3)$$
$$\Rightarrow 5y - 20 = x + 3$$
$$\Rightarrow 5y - x = 23 \leftarrow \text{General form of the equation.}$$

$\Rightarrow y - \frac{x}{5} = \frac{23}{5}$

$\Rightarrow y = \frac{1}{5} \cdot x + \frac{23}{5}$ ← Slope-intercept form where slope $= \frac{1}{5}$ and intercept $= \frac{23}{5}$.

(*) Two lines are said to be parallel if the two lines have the same slope.

(*) Two lines are said to be perpendicular if the product of their slops is (-1).

53. x intercept $= 2$; y intercept $= -1$. Find the equation of the line.

x intercept $= 2$ means the coordinate of the point, where the line intersects x axis is $(2, 0)$.

y intercept $= -1$ means the coordinate of the point, where the line intersects y axis is $(0, -1)$

Then the slope $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 0}{0 - 2} = \frac{-1}{-2} = \frac{1}{2}$

∴ The equation is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 0 = \frac{1}{2}(x - 2)$$

$$\Rightarrow y = \frac{1}{2}x - 1$$ ← slope-intersect form.

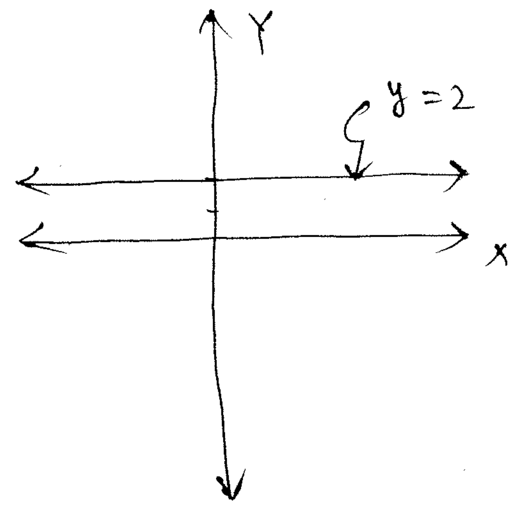
$$\Rightarrow 2y = x - 2$$

$$\Rightarrow 2y - x = -2$$ ← general form.

57. Horizontal containing the point $(-3,2)$

Slope of a horizontal line $= 0$

$\therefore$ The equation is



$$y-2 = 0\,(x+3)$$
$$\Rightarrow y-2=0$$
$$\Rightarrow y=2$$

59. Parallel to the line $y=2x$ containing the point $(-1,2)$

First calculate at least two points that are on the line $y=2x$. Obviously, two points are $(0,0)$, $(2,4)$

Calculate the slope of the line $y=2x$

$$m = \frac{4-0}{2-0} = 2$$

The line parallel to $y=2x$ has the same slope.

$\therefore$ The equation of the line parallel to $y=2x$ and containing the point $(-1,2)$ is

$$y-2 = 2\,(x+1)$$
$$\Rightarrow y-2 = 2x+2$$
$$\Rightarrow y-2x = 4 \quad \leftarrow \text{General form.}$$
$$\Rightarrow y = 2x+4 \quad \leftarrow \text{Slope - intercept form.}$$

66. Perpendicular to the line $y=2x-3$ containing the point $(1,-2)$

Calculate two points on the line $y=2x-3$, which are $(0,-3)$ and $(2,1)$

$$\therefore \text{Slope} = m = \frac{1+3}{2-0} = \frac{4}{2} = 2$$

As, The slope of the line perpendicular to $y = 2x - 3$ $\times$ the slope of the line $y = 2x - 3$ $= -1$

Then

$$m_1 \times m_2 = -1$$

when, m_2 = known = 2

$$\therefore \quad m_1 \times 2 = -1$$

$$\Rightarrow m_1 = -\frac{1}{2}$$

$\therefore$ So, the line perpendicular to $y = 2x - 3$ passing through the point $(1, -2)$ is

$$y + 2 = -\frac{1}{2}(x - 1)$$

$$\Rightarrow y + 2 = -\frac{x}{2} + \frac{1}{2}$$

$$\Rightarrow y + \frac{x}{2} = \frac{1}{2} - 2$$

$$\Rightarrow \frac{2y + x}{2} = \frac{1 - 4}{2}$$

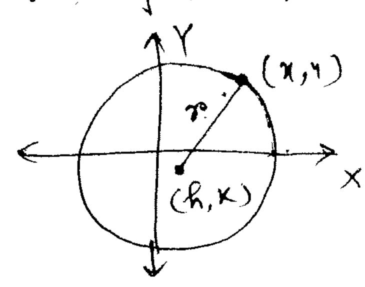
$$\Rightarrow x + 2y = -3 \quad \leftarrow \text{General form.}$$

$$\Rightarrow \frac{x}{2} + y = -\frac{3}{2}$$

$$\Rightarrow y = -\frac{x}{2} - \frac{3}{2} \quad \leftarrow \text{Slope-intercept form.}$$

2.4 Circles

A circle is a set of points in the xy plane that are a fixed distance r from a fixed point (h, k). The fixed distance r is called the radius, and the fixed point (h, k) is called the center of the circle.



let (x,y) represent the coordinates of any point on a circle with radius r and center (h,k). The distance between these two points (x,y) and (h,k) is therefore always equal to r. i.e.

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = r$$

$$\Rightarrow \sqrt{(x-h)^2+(y-k)^2} = r$$

$$\Rightarrow \boxed{(x-h)^2+(y-k)^2 = r^2}$$ ← General equation of a circle / standard form of a circle

If the ~~origin~~ center of the circle is at the origin, then, $(h,k) = (0,0)$

$$\therefore \boxed{x^2+y^2=r^2}$$ ← Equation of a circle with center at the origin $(0,0)$.

If the radius of the circle $= 1$, then the equation of a unit circle is

$$\boxed{x^2+y^2=1}$$ ← Equation of a unit circle with center at the origin $(0,0)$

let us now simplify the general form of a circle

$$(x-h)^2+(y-k)^2 = r^2$$

$$\Rightarrow x^2-2hx+h^2+y^2-2yk+k^2=r^2$$

$$\Rightarrow x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$$

$$\Rightarrow \boxed{x^2+y^2+ax+by+c=0}$$ ← Most general form of a circle / General form of a circle.

where $a=-2h$, $b=-2k$ and $c=(h^2+k^2-r^2)$.

18. Write the standard form of the equation and the general form whose radius is $r = 7$ and origin is at $(h, k) = (-5, -2)$

Standard form :

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\Rightarrow (x+5)^2 + (y+2)^2 = 7^2$$

$$\Rightarrow (x+5)^2 + (y+2)^2 = 49 .$$

General form :

$$x^2 + y^2 + ax + by + c = 0$$

where, $a = -2h$, $b = -2k$ and $c = (h^2 + k^2 - r^2)$

$\Rightarrow$ $a = -2\times(-5)$, $b = -2\times(-2)$, $c = 25 + 4 - 49$

$\Rightarrow$ $a = 10$, $b = 4$, $c = -20$

$\therefore$ The general form is

$$x^2 + y^2 + 10x + 4y - 20 = 0 .$$

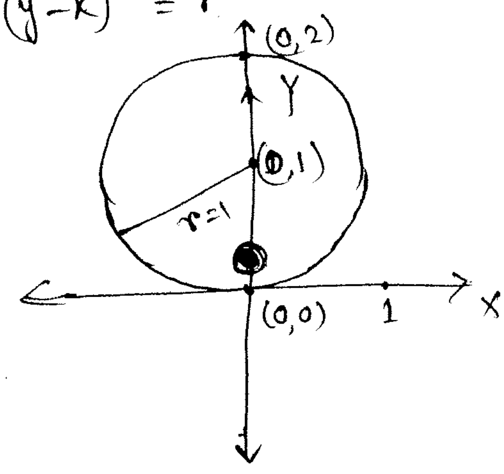
22 (a) Find the center (h, k) and radius of each circle (b) Graph each circle (c) Find the intercepts if any.

$$x^2 + (y-1)^2 = 1$$

Compare with the standard form $(x-h)^2 + (y-k)^2 = r^2$

$\therefore$ $h = 0$, $k = 1$, $r = 1$

$\therefore$ Center $= (h, k) = (0, 1)$ and radius $= 1$



x-intercept : Put $y = 0$

$$x^2 + (y-1)^2 = 1$$

$$\Rightarrow x^2 + (-1)^2 = 1$$

$$\Rightarrow x^2 = 0$$

$$\Rightarrow x = 0$$

y-intercept : Put $x = 0$

$$\Rightarrow x^2 + (y-1)^2 = 1$$

$$\Rightarrow (y-1)^2 = 1$$

$$\Rightarrow y-1 = \pm 1$$

1 $\Rightarrow y - 1 = +1$ $\Rightarrow y = 2$ | 2 $\Rightarrow y - 1 = -1$ $\Rightarrow y = 0$

$\therefore$ the intercepts are at $y = 0$ and $y = 2$

31. $2x^2 + 2y^2 - 12x + 8y - 24 = 0$

$$\Rightarrow x^2 + y^2 - 6x + 4y - 12 = 0$$

Compare with the general form

$$x^2 + y^2 + ax + by + c = 0$$

$\therefore a = -6$ $\Rightarrow -2h = -6$ $\Rightarrow h = 3$ | $b = 4$ $\Rightarrow -2k = 4$ $\Rightarrow k = -2$ | $c = -12$ $\Rightarrow h^2 + k^2 - r^2 = -12$ $\Rightarrow 9 + 4 - r^2 = -12$ $\Rightarrow -r^2 = -12 - 13$ $\Rightarrow r^2 = 25$ $\Rightarrow r = 5$ (as $r = -5$ is not allowed, because radius of a circle cannot be '0')

$\therefore$ center $= (h, k) = (3, -2)$

radius $= 5$ unit.

<u>x-intercept</u>: $(y=0)$

$$x^2 + ax + c = 0$$

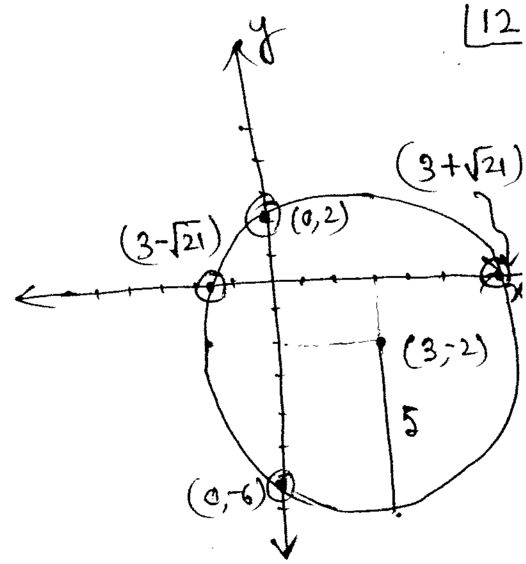
$$\Rightarrow x^2 - 6x - 12 = 0$$

$$x = \frac{6 \pm \sqrt{36+48}}{2}$$

$$= \frac{6 \pm \sqrt{84}}{2} \qquad 84 = 2 \times 2 \times 3 \times 7$$

$$= 3 \pm \sqrt{21}$$

$\therefore$ x intercepts are $(3+\sqrt{21})$ and $(3-\sqrt{21})$



<u>y-intercept</u>: $(x=0)$

$$y^2 + 4y - 12 = 0$$

$$\therefore y = \frac{-4 \pm \sqrt{16+48}}{2}$$

$$= \frac{-4 \pm \sqrt{64}}{2} \qquad 64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

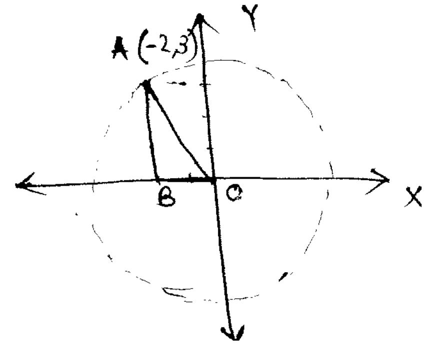
$$= \frac{-4 \pm 8}{2} = -2 \pm 4$$

$\therefore$ y intercepts are $2, -6$

35. Find the standard form of the equation of each circle
Center at the origin and containing the point $(-2,3)$

$\overline{AO}$ is the radius of the circle, which needs to be calculated.

$\overline{BO} = 2$ unit, $AB = 3$ unit



∴ By Pythagorean theorem

$AO^2 = BO^2 + AB^2$

$= 2^2 + 3^2$

$= 4 + 9$

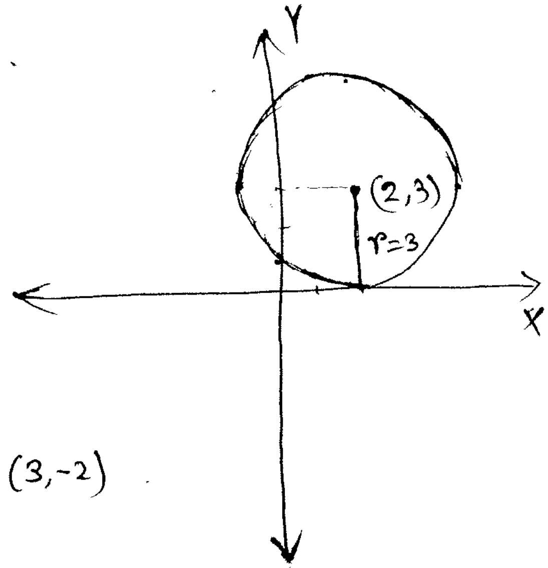
∴ $AO = \sqrt{13}$.

∴ The standard form of the equation of the circle

$(x-0)^2 + (y-0)^2 = (\sqrt{13})^2$

$\Rightarrow x^2 + y^2 = 13$

37. Center (2, 3) and tangent to the x-axis.

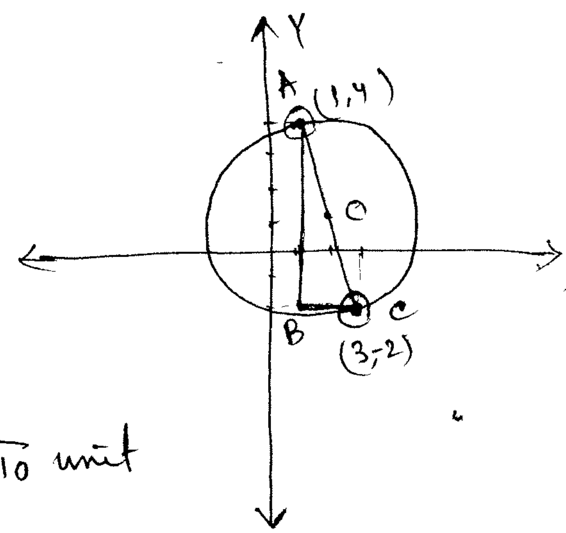


Equation:

$(x-2)^2 + (y-3)^2 = 3^2$

$\Rightarrow (x-2)^2 + (y-3)^2 = 9$

39. With endpoints of a diameter at (1, 4) and (3, −2).



$\overline{AB} = 6$ unit

$\overline{BC} = 2$ unit.

∴ $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$

$= 6^2 + 2^2$

$= 40$

∴ $\overline{AC} = \sqrt{40}$

∴ Radius $= \overline{AO} = \frac{1}{2}\overline{AC} = \frac{\sqrt{40}}{2} = \frac{2\sqrt{10}}{2} = \sqrt{10}$ unit

∴ According to midpoint formula, the center is at $\left(\frac{1+3}{2}, \frac{4-2}{2}\right) = (2, 1)$ ∴ Equation $= (x-2)^2 + (y-1)^2 = 10$