Lecture - 3   1.5 : Solving inequalities   21.01.2015   LL

- $a < b$: a is less than b.
- $a > b$: a is greater than b.
- $a \geq b$: a is greater or equals to b.
- $a \leq b$: a is less or equals to b.

Interval notation :

1. Open interval $\rightarrow$ Denoted by $(a, b) \rightarrow a < x < b$, which means if 'a' and 'b' are two real numbers, then all real numbers x consists of real numbers in the interval $a < x < b$.
2. Closed interval $\rightarrow$ Denoted by $[a, b] \rightarrow a \leq x \leq b$.
3. Half-open $\rightarrow$ Denoted by $(a, b] \rightarrow a < x \leq b$.
4. Half-closed $\rightarrow$ Denoted by $[a, b) \rightarrow a \leq x < b$.

Properties of inequalities :

1. For any real number a : $a^2 \geq 0$.
2. For real numbers a, b and c : If $a < b$, then $a \pm c < b \pm c$
   If $a > b$, then $a \pm c > b \pm c$.
3. For real numbers a, b and c : If $a < b$, then $ac < bc$
   If $a > b$, then $ac > bc$
   if and only if $c > 0$.
4. For real numbers a, b and c : If $a < b$, then $ac > bc$
   If $a > b$, then $ac < bc$
   if and only if $c < 0$.

5. If $a>0$, then $\frac{1}{a}>0$ and If $\frac{1}{a}>0$, then $a>0$

6. If $a<0$, then $\frac{1}{a}<0$ and If $\frac{1}{a}<0$, then $a<0$.

Problems:

52. If $-\frac{1}{4}x>1$ then x ___ -4. Fill up the blanck.

$-\frac{1}{4}x > 1$

or, $-\frac{1}{4}\cdot 4\cdot x > 4$ ← by Rule 3

or, $-x > 4$

or, $-x\cdot(-1) < 4\cdot(-1)$ ← by Rule 4, the inequality symbol is reversed.

or, $x < -4$

56. $2-3x \le 5$

$\Rightarrow 2-3x-2 \le 5-2$

$\Rightarrow -3x \le 3$

$\Rightarrow \frac{\cancel{-3}x}{\cancel{-3}} \ge \frac{3}{-3}$ (By Rule 4)

$\Rightarrow x \ge -1$

64. $8-4(2-x) \le -2x$

$\Rightarrow \cancel{8}-\cancel{8}+4x \le -2x$

$\Rightarrow 4x \le -2x$

$\Rightarrow 4x+2x \le -2x+2x$ (Rule 2)

$\Rightarrow 6x \le 0$

$\Rightarrow x \le 0$ (By Rule 3)

73. $-3 < \frac{2x-1}{4} < 0$

$\Rightarrow -3 \cdot 4 < \frac{2x-1}{\cancel{4}} \cdot \cancel{4} < 0 \cdot 4$ (By Rule 3)

$\Rightarrow -12 < 2x-1 < 0$

$\Rightarrow -12+1 < 2x-1+1 < 0+1$ (Rule 2)

$\Rightarrow -11 < 2x < 1$

$\Rightarrow \frac{-11}{2} < \frac{\cancel{2}x}{\cancel{2}} < \frac{1}{2}$ (Rule 3)

$\Rightarrow -\frac{11}{2} < x < \frac{1}{2}$

86. $0 < \frac{4}{x} < \frac{2}{3}$

$\Rightarrow 0 \times \frac{1}{4} < \frac{4}{x} \times \frac{1}{4} < \frac{2}{3} \times \frac{1}{4}$

$\Rightarrow 0 < \frac{1}{x} < \frac{1}{6}$

$\Rightarrow 0 < x < 6$ (Rule 5)

A non-technical summary:

1. The direction of the inequality does not change when –
   (i) Add or subtract a number from both sides
   (ii) Multiply or divide both sides by a positive number.
   (iii) Simplify a side.
2. The direction of the inequality changes when –
   (i) Multiply or divide both sides by a negative number.
   (ii) Swapping left and right hand side

Let us explain the exceptional case. That is the case when we multiply both sides by a negative number, the direction of the inequality changes. Though it looks very strange, it is the fact. Let us consider an example. Consider we have an inequality

$$-10 < -2$$

If one does not change the sign when multiplying both sides by (-1), the answer is

$$(-10)(-1) < (-2)(-1)$$

$$\Rightarrow \quad 10 < 2 \quad \Leftarrow \text{clearly this is not true.}$$

But if one flips the sign, the answer is correct.

$$(-10)(-1) > (-2)(-1)$$

$$\Rightarrow \quad 10 > 2 \quad \Leftarrow \underline{\text{Right answer}}.$$

- <u>A simple trick to check your solution</u>.

Consider an example:

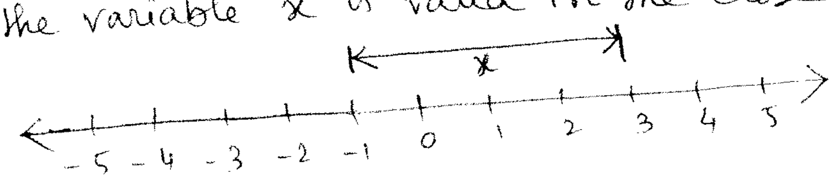
$$-3 \le 2x - 1 \le 5 \qquad \text{------ (1)}$$

$$\Rightarrow -3+1 \le 2x \le 5+1 \qquad \text{(Rule 2)}$$

$$\Rightarrow \frac{-2}{2} \le \frac{2x}{2} \le \frac{6}{2} \qquad \text{(Rule 3)}$$

$$\Rightarrow -1 \le x \le 3$$

That means the variable x is valid in the closed interval $[-1, 3]$



Now let us check our solution by choosing a value of x from the allowed interval.

Consider $x = 1$ and put $x = 1$ in the equation number ①,

$$-3 \le 2(1) - 1 \le 5$$

$\Rightarrow -3 \le 1 \le 5$ ← This is of course true.

Now consider $x = -1$,

$$-3 \le 2\cdot(-1) - 1 \le 5$$

$\Rightarrow -3 \le -3 \le 5$ ← This is also true.

Now we consider another value of x, which is outside of the allowed range. Say $x = -2$

Then $-3 \le 2(-2) - 1 \le 5$

$\Rightarrow -3 \le -5 \le 5$ ⇐ Wrong.

That confirms that your solution is correct.

1.6 : Equations and inequalities involving absolute value

If a is a positive real number and if u is any algebraic expression, then

$|u| = a$ is equivalent to $u = a$ or $u = -a$.

Problem - 9

$$|2x + 3| = 5$$

In this case, we have two possibilities.

① $\Rightarrow 2x + 3 = 5$

$\Rightarrow 2x = 5 - 3$

$\Rightarrow x = \frac{5-3}{2}$

$\Rightarrow x = 1$

② $\Rightarrow 2x + 3 = -5$

$\Rightarrow 2x = -5 - 3$

$\Rightarrow x = \frac{-8}{2}$

$\Rightarrow x = -4$

∴ The solution is $x = \{1, -4\}$

12. $|1-2z|+6=9$

$\Rightarrow |1-2z| = 9-6$

$\Rightarrow |1-2z| = 3$

① $\Rightarrow 1-2z = 3$

$\Rightarrow -2z = 3-1$

$\Rightarrow -2z = 2$

$\Rightarrow z = \frac{2}{-2}$

$\Rightarrow z = -1$

② $\Rightarrow 1-2z = -3$

$\Rightarrow -2z = -3-1$

$\Rightarrow -2z = -4$

$\Rightarrow z = \frac{-4}{-2}$

$\Rightarrow z = 2$

$\therefore$ The solution is $z = \{-1, 2\}$.

26. $|x^2-16| = 0$

① $\Rightarrow x^2-16 = +0$

$\Rightarrow (x)^2-(4)^2 = 0$

② $\Rightarrow x^2-16 = -0$

$\Rightarrow (x)^2-(4)^2 = 0$

That means the two equations are same, so we can consider any one of them.

$(x)^2-(4)^2 = 0$

$\Rightarrow (x+4)(x-4) = 0$

$a^2-b^2 = (a+b)(a-b)$

Either, $x+4=0$

$\Rightarrow x = -4$

or, $x-4=0$

$\Rightarrow x = 4$

$\therefore$ The solution is x is either 4 or (-4).

30. $|x^2+3x-2| = 2$

① $\Rightarrow x^2+3x-2=2$

$\Rightarrow x^2+3x-4=0$

Quadratic formula

$$x = \frac{-3 \pm \sqrt{3^2-4(1)(-4)}}{2\cdot 1}$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{25}}{2}$$

$$\Rightarrow x = \frac{-3 \pm 5}{2}$$

$$\therefore x = \frac{-3+5}{2}, \frac{-3-5}{2}$$

$\therefore x = 1, -4$

② $\Rightarrow x^2+3x-2=-2$

$\Rightarrow x^2+3x=0$

$\Rightarrow x(x+3)=0$

Either, $x=0$

or, $x+3=0$

$\Rightarrow x=-3$.

$\therefore$ Solutions are $x = 0, 1, -3, -4$.

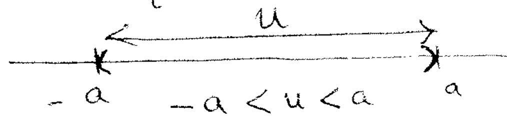
Solving inequalities involving absolute value

- If a is a positive number and if u is an algebraic expression, then

$|u| < a$ is equivalent to $-a < u < a$

$|u| \le a$ is equivalent to $-a \le u \le a$.

In other words, $|u| < a$ is equivalent to $-a < u$ and $u < a$.



To say more clearly.

$|u| < a$ means $\pm u < a$

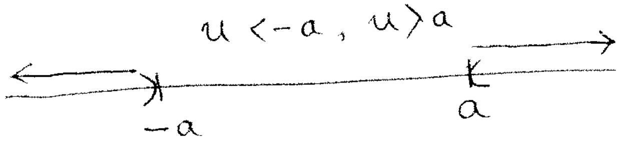
That means $u < a$ ~~and~~ and $-u < a$

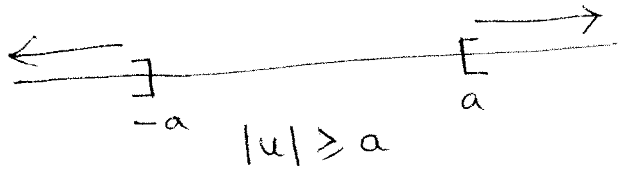
$\Rightarrow u > -a$

- If a is a positive number and u is an algebraic expression, then

$|u| > a$ is equivalent to $u < -a$ or $u > a$

$|u| \geq a$ is equivalent to $u \leq -a$ or $u \geq a$.





To say more clearly, $|u| > a$ means $\pm u > a$

That means $u > a$ or $-u > a$

$\Rightarrow u < -a$

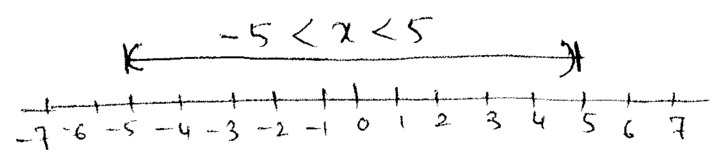
Problems

36. $|3x| < 15 \Rightarrow \pm 3x < 15$

$\therefore$ $3x < 15$ and $-3x < 15$

$\Rightarrow$ $x < 5$ and $x > -5$

$\therefore$ $-5 < x < 5$



41. $|3t - 2| \leq 4$

$\Rightarrow \pm(3t - 2) \leq 4$

$\therefore$ $(3t - 2) \leq 4$ and $-(3t - 2) \leq 4$

$\Rightarrow 3t \leq 6$ $\qquad \Rightarrow (3t - 2) \geq -4$

$\Rightarrow t \leq 2$ $\qquad \Rightarrow 3t \geq -2$

$\Rightarrow t \geq -\frac{2}{3}$

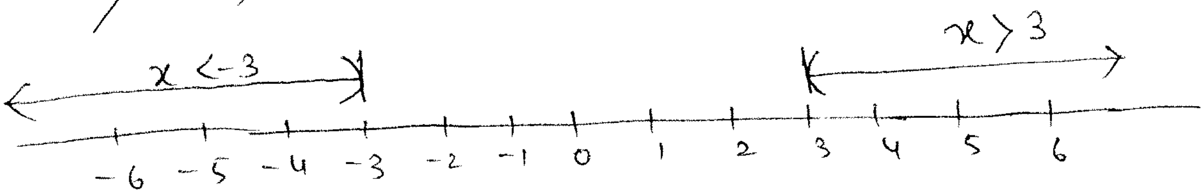
$\therefore$ $\underline{-\frac{2}{3} \leq t \leq 2}$.

38. $|2x| > 6$

$\Rightarrow \pm 2x > 6$

$\therefore\ 2x > 6$ and $\Rightarrow -2x > 6$

$\Rightarrow x > 3$ and $\Rightarrow x < -3$



43. $|2x-3| \geq 2$

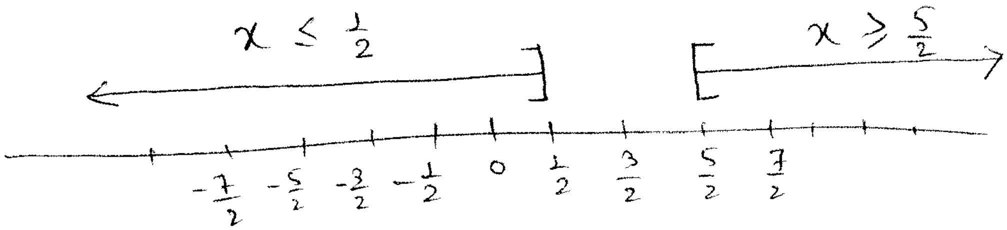
$\Rightarrow \pm(2x-3) \geq 2$

$\therefore\ 2x-3 \geq 2$ and $-(2x-3) \geq 2$

$\Rightarrow 2x \geq 5$ and $\Rightarrow 2x-3 \leq -2$

$\Rightarrow x \geq \frac{5}{2}$ and $\Rightarrow 2x \leq 1$

$\Rightarrow x \leq \frac{1}{2}$



1.7 : Problem solving : Interest, Mixtures, Uniform motion and constant rate Job applications.

<u>Interest</u> : Interest is money paid for the use of money.

<u>Principal</u> : The total amount of money borrowed from a bank in the form of a loan or by any other means.

<u>Rate of interest</u> · expressed as a percent.

Simple interest:

If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged as

$$I = P \cdot r \cdot t$$

Example: Suppose that Julia borrows \$1500 for 1 year and 6 months at the simple interest rate of 11% per annum. What is the interest that Julia will be charged on the loan? How much does Julia owe after 6 months.

Principal $P = \$1500$

Interest rate $r = 11\% = 11 \div 100 = 0.11$.

Time $t = 1$ year and 6 months. $= 1\frac{1}{2}$ years.

$\therefore$ Interest charged $I = P.r.t$

$= (1500) \times (0.11) \times \frac{3}{2}$

$= \$ 247.50$

After ~~one~~ 1 year and 6 monts Julia owe what she borrowed from the bank plus the interest:

$$\$1500 + \$247.50 = \$1747.50$$

17. Betsy, a recent retiree, requires \$6000 per year in extra income. She has \$50,000 to invest and can invest in B-rated bonds paying 15% per year or in a certificate of deposit (CD) paying 7% per year. How much money should be invested in each to realize exactly \$6000 in interest per year?

Principal $P = \$50000$

Time $t = 1$ year

Interest rate of bonds $= 15\%$ and of certificate deposit $= 7\%$.

let us consider that Betsy requires $\$x$ to invest to the bonds, where $\$x$ is an unknown amount needs to solve. Therefore, it is clear that Betsy has to ~~requires~~ $\$(50000-x)$ to invest to the certificate deposit.

$\therefore$ Principal for bonds $= \$x$, interest rate $= 15\%$
whereas, Principal for CD $= \$(50000-x)$, interest rate $= 7\%$

Since the total interest has to be $\$6000$, one needs to solve the following equation:

Interest from Bonds + Interest from CD = 6000

$$\Rightarrow (x \times 0.15 \times 1) + (50000 - x) \times 0.07 \times 1 = 6000$$

$$\Rightarrow 0.15x + (50000 \times 0.07) - 0.07x = 6000$$

$$\Rightarrow 0.08x = 6000 - (50000 \times 0.07)$$

$$\Rightarrow x = \frac{6000 - 3500}{0.08} = \frac{2500}{0.08}$$

$$= 31250$$

So, Betsy requires to invest $\$31250$ to the bonds and the rest $(50000 - 31250 = \$17750)$ to the certificate deposit to earn exactly $\$6000$ in interest per year.

Solve mixture problems:

21. The manager of a store that specializes in selling tea decides to experiment a new blend. She will mix some earl Grey tea that sells for $\$5$ per pound with some orange pekoe tea that sells for $\$3$ per pound to get 100 pounds of the new blend. The selling price of the new blend is to be $\$4.50$ per pound, and there is no difference in revenue

from selling the new blend verses selling the other types. How many pounds of the Earl Grey and Orange Pekoe tea are required?

let us consider that manger needs to mix x pounds of earl grey tea. Obviously she requires $(100-x)$ pounds of Orange Pekoe tea to mix with Earl grey tea to obtain a blend of 100 pounds.

One therefore requires to solve

$$5 \times x + (100-x)\,3 = 4.5 \times 100$$

$$\Rightarrow 5x + 300 - 3x = 450$$

$$\Rightarrow 2x = 450 - 300$$

$$\Rightarrow x = \frac{150}{2}$$

$$\Rightarrow x = 75$$

So, she requires 75 pounds of earl grey tea and $(100-75)$ $= 25$ pounds of Orange Pekoe tea to obtain a blended tea of 100 pounds which sells at a rate of \$4.50 per pound.

<u>Solve uniform motion problems</u>:

Objects that move at a constant speed are said to be in uniform motion. When the average speed of an object is known, it can be interopreted as its constant speed.

If an object moves at an average speed (rate) r, the distance d covered in time t is given by the formula $\longrightarrow d = rt$, equivalently the average speed of an object covering a distance d at time t is given by $\rightarrow r = \frac{d}{t}$

28. Two cars enter the Florida Turnpike at Commercial Boulevard at 8:00 AM, each heading for Wildwood. One car's average speed is 10 miles per hour more than the other's. The faster car arrives at Wildwood at 11:00 AM; $\frac{1}{2}$ hour before the other car. What was the average speed of each car? How far did each travel?

Consider the faster car's average speed is x miles/hour.

The the slower car's average speed is $(x-10)$ miles/hour.

The faster car takes 2 hours to reach the destination.

The distance covered by the faster car moving at a speed of x miles/hour is given by the formula

$$d = r.t$$
$$= x \cdot 2 = 2x$$

The slower car takes $2\frac{1}{2}$ hours to reach the same destination, which means both cars are travelling same distance but with different speed.

The distance covered by the slower car moving at a speed of $(x-10)$ miles/hous is therefore

$$d = r.t$$
$$= (x-10) \times 2\frac{1}{2} = (x-10) \times \frac{5}{2}.$$

Since the two cars covering same distance, one can clearly say

$$2x = (x-10) \times \frac{5}{2}$$
$$\Rightarrow 2x = \frac{5}{2}x - 25$$
$$\Rightarrow \frac{5}{2}x - 2x = 25$$

$$\Rightarrow \left(\frac{5}{2}-2\right)x = 25$$

$$\Rightarrow \frac{5-4}{2}x = 25$$

$$\Rightarrow \frac{1}{2}x = 25$$

$$\Rightarrow x = 50.$$

Therefore speed of the faster car is 50 miles/hour.
and of the slower car is $(50-10) = 40$ miles/hour.

Each car travels a distance of $2x = (x-10)\frac{5}{2}$
$= 100$ miles.

<u>25</u>. A motorboat can maintain a constant speed of 16 miles per hour relative to the water. The boat makes a trip upstream to a certain point in 20 minutes; the return trip takes 15 minutes. What is the speed of the current?

Speed of the motorboat $= 16$ miles/hour.
Consider, speed of the current $= x$ miles/hour.
The distance covered during an upstream trip is

$$d = r.t$$
$$= (16-x)\times\frac{1}{3}$$
$$= \frac{(16-x)}{3}\text{miles}.$$

60 minutes $= 1$ hour
$\therefore$ 1 minutes $= \frac{1}{60}$ hour
$\therefore$ 20 minutes $= \frac{20}{60} = \frac{1}{3}$ hour

The distance covered during the downstream trip is

$$d = r.t$$
$$= (16+x)\times\frac{1}{4} = \frac{16+x}{4}\text{ miles}.$$

Eventually, same distance is covered for the upstream and the return trip. One therefore needs to solve

$$\frac{16-x}{3} = \frac{16+x}{4}$$

$$\Rightarrow 4(16-x) = (16+x)3$$

$$\Rightarrow 64-4x = 48+3x$$

$$\Rightarrow 4x+3x = 64-48$$

$$\Rightarrow 7x = 16$$

$$\Rightarrow x = \frac{16}{7}$$

$\therefore$ Speed of the current is $\frac{16}{7}$ miles / hour.

Solve constant rate job problems

33. Trent can deliver his newspapers in 30 minutes. It takes Lois 20 minutes to do the same route. How long would it take them to deliver the newspaper if they work together?

Consider, working together the job could be done in t ~~hours~~ minutes.

In 1 minute Trent does $\frac{1}{30}$ part of the job, whereas in 1 minute Lois does $\frac{1}{20}$ part of the job.

$\therefore$ Total job completed in one minute by Trent and Lois together $= \frac{1}{30} + \frac{1}{20} = \frac{2+3}{60} = \frac{5}{60} = \frac{1}{12}$ th of the total job.

On the other hand, if one considers that the total job can be finished in t minutes when Trent and Lois work together, then the amount of job done in 1 minutes

$$= \frac{1}{t}$$

$$\therefore \quad \frac{1}{12} = \frac{1}{t}$$

$$\Rightarrow t = 12$$

That means, they could finish the whole work in 12 minutes, if they would work together.

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