The nth root of a number 'a', where 'n' is a positive integer is a number 'b' which when raised to the power 'n' yields 'a':

$$\sqrt[n]{a} = b \quad \text{means} \quad a = b^n$$

$$\sqrt[n]{a^m} = a^{m/n}$$

If n = even $(2, 4, 6, 8, \ldots)$ then $a \geq 0$ and $b \geq 0$

If n = odd $(3, 5, 7, 9 \ldots)$ then a, b = Real numbers

when $n = 2$: $\sqrt[2]{a} = \sqrt{a}$ = Square root of a.

when $n = 3$: $\sqrt[3]{a}$ = Cube root of a.

when $n = 4$: $\sqrt[4]{a}$ = Fourth root of a.

Examples:

1. n is an odd number, $n \geq 3$: $\sqrt[n]{a^n} = a$

$$\sqrt[5]{a^5} = a, \quad \sqrt[3]{a^3} = a, \quad \sqrt[3]{125} = \sqrt[3]{5^3} = 5, \quad \sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$$

2. n is an even number, $n \geq 2$: $\sqrt[n]{a^n} = |a|$

$$\sqrt[4]{16} = \sqrt[4]{2^4} = |2| = 2 \; ; \quad \sqrt[4]{(-2)^4} = |-2| = 2$$

Identities and properties:

1. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Rationalizing Denominators:

$\cancel{\dfrac{\sqrt{2}}{\sqrt{3}-3\sqrt{2}}}$

49. $$\frac{2-\sqrt{5}}{2+3\sqrt{5}} = \frac{2-\sqrt{5}}{2+3\sqrt{5}} \cdot \frac{2-3\sqrt{5}}{2-3\sqrt{5}} = \frac{(2-\sqrt{5})(2-3\sqrt{5})}{(2)^2-(3\sqrt{5})^2}$$

$$= \frac{4-6\sqrt{5}-2\sqrt{5}+15}{4-45} = \frac{19-8\sqrt{5}}{-41} = \frac{8\sqrt{5}-19}{41}.$$

Rationalizing denominator means removing roots from the denominator.

Domain of a variable :

$\frac{10}{0} = \frac{x}{0} = \frac{2}{0} = -\frac{5}{0} =$ Not defined / Not allowed.

The set of values of a variable is called the domain of the variable if the variable is defined in that range.

- $\frac{10}{y-5}$ the domain of whis is $\{y \mid y \neq 5\}$.

  For $y = 5$, $\frac{10}{y-5}$ = Not defined

  For $y \neq 5$, $\frac{10}{y-5}$ is defined

- If c is the circumference of a circle of radius r then $c = 2\pi r$. As radius and circumference of a circle can not be negative, the domain of r is therefore set of poselive real numbers. One can represent them in rostar notation:

  The domain of r is $\{r \mid r$ = Poselive real numbers$\}$

  $$= \{r \mid r = \mathbb{R}^+\}$$

  The domain of c is $\{c \mid c = \mathbb{R}^+\}$

1.1 Linear equations

A linear equation is an algebraic equation in which each term is either a constant or a product of a constant and a single variable (of power 1).

$ay = b$: Linear equation in one variable y, where $a = b =$ Constant.

$y = mx + c$: linear equation in two variables x and y where $m = c =$ constant.

Nonlinear equations:

1. $xy + 5 = a$, where a = Constant.
2. $y^2 + 5x = 2$
3. $x^{1/3} + 5 = -2$
4. $5 \sin x + a = b$, where $a = b =$ Constant.

etc.

To solve an ~~linear~~ equation means finding all the solutions of that ~~linear~~ equation. Note that there might have more than one solution of a single equation.

Examples:

17. $3x + 4 = x$

or, $3x + 4 - x = x - x$

or, $2x + 4 = 0$

or, $2x + 4 - 4 = 0 - 4$

or, $2x = -4$

or, $\frac{2x}{2} = \frac{-4}{2}$

or, $x = -2$

* Notice the symbol 'or' at the begining of each line. One can not put an equality sign for solving an equation.

* One can use either the symbol 'or' or '$\Rightarrow$'

60. $\frac{-4}{2x+3} + \frac{1}{x-1} = \frac{1}{(2x+3)(x-1)}$

* Least common multiplier of $(2x+3)$ and $(x-1)$ is $(2x+3)(x-1)$

$\Rightarrow \frac{-4(x-1) + 1(2x+3)}{(2x+3)(x-1)} = \frac{1}{(2x+3)(x-1)}$

$\Rightarrow \frac{-4x + 4 + 2x + 3}{\cancel{(2x+3)}\cancel{(x-1)}} \cancel{(2x+3)}\cancel{(x-1)} = \frac{1}{\cancel{(2x+3)}\cancel{(x-1)}} \cancel{(2x+3)}\cancel{(x-1)}$

$\Rightarrow -2x + 7 = 1 \quad \Rightarrow -2x + 7 - 7 = 1 - 7 \quad \Rightarrow -2x = -6$

$\Rightarrow \frac{+2x}{+2} = +6/+2$

$\Rightarrow x = 3$

64. $$\frac{x+1}{x^2+2x} - \frac{x+4}{x^2+x} = \frac{-3}{x^2+3x+2}$$

L.C.M. of (x^2+2x) and x^2+x :

$$\left.\begin{aligned} x^2+2x &= \check{x}(x+2) \\ x^2+x &= \check{x}(x+1) \end{aligned}\right\} \quad x(x+1)(x+2)$$

$$\Rightarrow \frac{x+1}{x(x+2)} - \frac{x+4}{x(x+1)} = \frac{-3}{x^2+3x+2}$$

$$\Rightarrow \frac{(x+1)(x+1) - (x+4)(x+2)}{x(x+1)(x+2)} = \frac{-3}{x^2+3x+2}$$

$$\Rightarrow \frac{x^2+2x+1-(x^2+2x+4x+4)}{x(x+1)(x+2)} = \frac{-3}{x^2+2x+x+2}$$

$$\Rightarrow \frac{\cancel{x^2}+2x+1-\cancel{x^2}-6x-4}{x(x+1)(x+2)} = \frac{-3}{x(x+2)+1(x+2)}$$

$$\Rightarrow \frac{-4x-3}{x(x+1)(x+2)} = \frac{-3}{(x+2)(x+1)}$$

$$\Rightarrow \frac{-4x-3}{x\cancel{(x+1)}\cancel{(x+2)}}\cancel{(x+1)}\cancel{(x+2)} = \frac{-3}{\cancel{(x+2)}\cancel{(x+1)}}\cancel{(x+2)}\cancel{(x+1)}$$

$$\Rightarrow \frac{-4x-3}{x} = \frac{-3}{1} \qquad \left[\because \frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc\right.$$

$$\Rightarrow -4x-3 = -3x$$

$$\Rightarrow -4x-3+3x = -3x+3x$$

$$\Rightarrow -x-3+3 = 3$$

$$\Rightarrow x = -3$$

1.2 Quadratic equation

A quadratic equation is an equation having the form :

$$ax^2 + bx + c = 0$$

, where a, b and c are the coefficients of the equation,

$a = b = c$ = Constant.

If $a = 0$, then the equation becomes linear equation

Solving a quadratic equation by quadratic formula

The solution of $ax^2 + bx + c = 0$ is :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$(b^2 - 4ac)$ is called the discriminant of the equation. The value of the discriminant tells us whether the equation has real solutions.

1. If $(b^2 - 4ac) > 0$, there are two unequal real solutions.
2. If $(b^2 - 4ac) = 0$, there are two equal solutions.
3. If $(b^2 - 4ac) < 0$, there is no real solution, all the solutions are imaginary.

Problems :

27. $$\frac{4(x-2)}{x-3} + \frac{3}{x} = \frac{-3}{x(x-3)}$$

$$\Rightarrow \frac{4x(x-2) + 3(x-3)}{x(x-3)} = \frac{-3}{x(x-3)}$$

$$\Rightarrow 4x^2 - 8x + 3x - 9 = -3$$

$\cancel{\Rightarrow 4x^2 - 5x - 14 = 0}$ ~~← Quadratic equation.~~

$\Rightarrow 4x^2 - 5x - 6 = 0$ ⇐ Quadratic equation

$\Rightarrow 4x^2 - 8x + 3x - 6 = 0$

$\Rightarrow 4x(x-2) + 3(x-2) = 0$

$\Rightarrow (x-2)(4x+3) = 0$

Either $x - 2 = 0$

$\Rightarrow x = 2$

or, $4x + 3 = 0$

$\Rightarrow 4x = -3$

$\Rightarrow x = \frac{-3}{4}$

$\therefore$ The solutions are

$x = 2, -\frac{3}{4}$.

$A = 4,\ B = -5,\ C = -6$

$AC = -24$

$$\begin{array}{r|l} 2 & 24 \\ \hline 2 & 12 \\ \hline 2 & 6 \\ \hline & 3 \end{array}$$

$\therefore 24 = 2 \times 2 \times 2 \times 3 = 8 \times 3$

One can then try all the possible combinations from the result.

one can choose $a = -8$ and $b = 3$

so that $ab = -24$ and $a + b = -5$

34. $(3z-2)^2 = 4$

$\Rightarrow \sqrt{(3z-2)^2} = \sqrt{4}$

$\Rightarrow 3z - 2 = \pm\sqrt{4}$

$\Rightarrow 3z - 2 = \pm 2$

$\therefore$ Either, $3z - 2 = 2$

$\Rightarrow 3z = 4$

$\Rightarrow z = \frac{4}{3}$

or, $3z - 2 = -2$

$\Rightarrow 3z = 0$

$\Rightarrow z = \frac{0}{3} = 0$.

$\therefore$ The solutions are $z = 0, \frac{4}{3}$.

37. $x^2 - \frac{x}{2} - \frac{3}{16} = 0$

$\Rightarrow \frac{16x^2 - 8x - 3}{16} = 0$

$\Rightarrow 16x^2 - 8x - 3 = 0$

$\Rightarrow 16x^2 - 8x + 1 - 1 - 3 = 0$

$\Rightarrow 16x^2 - 8x + 1 = 4$

$\Rightarrow 16x^2 - 4x - 4x + 1 = 4$

$\Rightarrow 4x(4x-1) - 1(4x-1) = 4$

$\Rightarrow (4x-1)(4x-1) = 4$

$\Rightarrow (4x-1)^2 = 4$

$\Rightarrow 4x - 1 = \pm\sqrt{4} = \pm 2$

Either, $4x - 1 = +2$ or, $4x - 1 = -2$

$\Rightarrow 4x = 3$ $\Rightarrow 4x = -1$

$\Rightarrow x = \frac{3}{4}$ $\Rightarrow x = -\frac{1}{4}$.

$\therefore$ The solutions are $x = \frac{3}{4}, -\frac{1}{4}$.

64. $\frac{2x}{x-3} + \frac{1}{x} = 4$

$\Rightarrow \frac{2x^2 + x - 3}{x(x-3)} = 4$

$\Rightarrow 2x^2 + x - 3 = 4x(x-3)$

$\Rightarrow 2x^2 + x - 3 = 4x^2 - 12x$

$\Rightarrow -2x^2 + 13x - 3 = 0$

$\therefore$ $A = -2$, $B = 13$, $C = -3$

$$\therefore \quad x = \frac{-13 \pm \sqrt{(13)^2 - 4(-2)(-3)}}{2\cdot(-2)} \qquad \Bigg| \qquad x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-13 \pm \sqrt{145}}{-4}$$

Either, $x = \dfrac{-13+\sqrt{145}}{-4}$ or, $x = \dfrac{-13-\sqrt{145}}{-4}$

$\Rightarrow x = \dfrac{13-\sqrt{145}}{4}$ $\qquad \Rightarrow x = \dfrac{13+\sqrt{145}}{4}$.

1.4 Solving radical equations :

31. $\sqrt{3-2\sqrt{x}} = \sqrt{x}$

$\Rightarrow 3 - 2\sqrt{x} = x$

$\Rightarrow 3 - x = 2\sqrt{x}$

$\Rightarrow (3-x)^2 = (2\sqrt{x})^2$

$\Rightarrow 9 + x^2 - 6x = 4x$

$\Rightarrow x^2 - 4x - 6x + 9 = 0$

$\Rightarrow x^2 - 10x + 9 = 0$

$\Rightarrow x^2 - 9x - x + 9 = 0$

$\Rightarrow x(x-9) - 1(x-9) = 0$

$\Rightarrow (x-9)(x-1) = 0$

Either, $x - 9 = 0$ or, $x - 1 = 0$

$\Rightarrow x = 9$ $\qquad \Rightarrow x = 1$.

40. $x^{3/4} - 9x^{1/4} = 0$

$\Rightarrow x^{3/4} = 9x^{1/4}$

$\Rightarrow \left(x^{3/4}\right)^4 = \left(9x^{1/4}\right)^4$

$\Rightarrow x^3 = 9x$

$\Rightarrow x^2 = 9$

$\Rightarrow x^2 - 9 = 0$

$\Rightarrow (x+3)(x-3) = 0$

$\therefore x = 3, -3$.

62. $\sqrt[4]{4-5x^2} = x$

$\Rightarrow (4-5x^2)^{1/4} = x$

$\Rightarrow (4-5x^2)^{\frac{1}{4}\times 4} = x^4$

$\Rightarrow (4-5x^2) = x^4$

$\Rightarrow -x^4 - 5x^2 + 4 = 0$

$\Rightarrow x^4 + 5x^2 - 4 = 0$

Consider $x^2 = u$, then

$u^2 + 5u - 4 = 0$

Quadratic formula:

$$u = \frac{-5 \pm \sqrt{25+16}}{2} = \frac{-5 \pm \sqrt{41}}{2}$$

No real roots.

42. $x^4 - 10x^2 + 25 = 0$

$$\therefore x^2 = \frac{10 \pm \sqrt{(10)^2 - 4.25.1}}{2} = \frac{10 \pm \sqrt{0}}{2} = 5$$

$\therefore x = \pm\sqrt{5}$

$\therefore$ The equation has two real roots $\{+\sqrt{5}, -\sqrt{5}\}$.