(A)

MAT 2322 C
Calculus III
Midterm # 2
March 21, 2013

Length: 80 minutes

Professor: Steve Desjardins

NAME Solutions

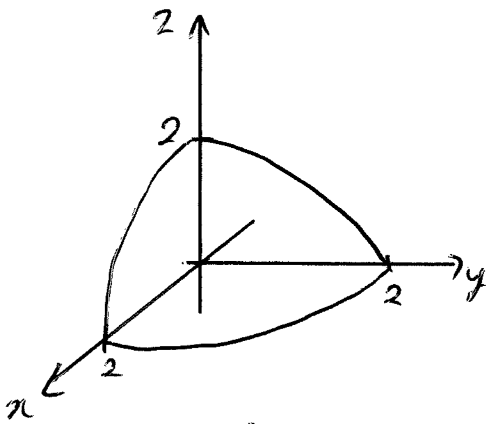
I.D.#

Instructions:

- Print your name and student number on this page.
- Verify that your copy of the midterm has exactly 6 pages (including this one).
- There are 5 questions worth 4 marks each for a total of 20 points.
- You must answer all questions.
- Write your answers in the space provided below each question. You may use the backs of pages if necessary.
- **You are not allowed to use any books or notes.**
- **You may use a basic scientific calculator. Graphing or programmable calculators are not permitted.**

(A)

1. Evaluate $\iiint_E 6e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where E is the region inside the sphere $x^2+y^2+z^2=4$ in the first octant.



first octant, so $0 \le \phi \le \pi/2$, $0 \le \theta \le \pi/2$

$$\iiint_E 6e^{(x^2+y^2+z^2)^{3/2}}\, dV$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} 6\, e^{\rho^3} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\pi/2} \sin\phi\, d\phi\right)\left(\int_0^{2} 6\rho^2 e^{\rho^3} d\rho\right)$$

$$= (\pi/2)\left(-\cos\phi \Big|_0^{\pi/2}\right)\left(2e^{\rho^3}\Big|_0^{2}\right)$$

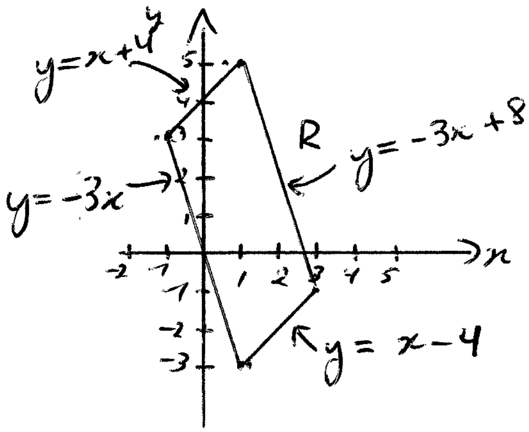
$$= (\pi/2)(1)(2(e^8-1))$$

$$= \boxed{\pi(e^8-1)}$$

$$\simeq \boxed{9362}$$

A

2. Evaluate $\displaystyle\iint_R (8x+4y)\,dA$, where R is the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$ and $(1,5)$ using the change of variable $x = \frac{1}{4}(u+v)$, $y = \frac{1}{4}(v-3u)$.



$y=-3x$ $\qquad \frac{1}{4}(v-3u) = -\frac{3}{4}(u+v)$

$v-3u = -3u-3v$

$4v = 0 \Rightarrow \boxed{v=0}$

$y = x+4$ $\qquad \frac{1}{4}(v-3u) = \frac{1}{4}(u+v)+4$

$v-3u = u+v+16$

$\boxed{u=-4}$

$y = x-4$ $\qquad \frac{1}{4}(v-3u) = \frac{1}{4}(u+v)-4$

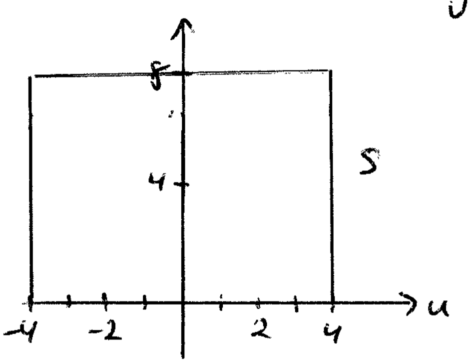
$v-3u = u+v-16$

$\boxed{u=4}$

$y=-3x+8$ $\qquad \frac{1}{4}(v-3u) = -\frac{3}{4}(u+v)+8$

$v-3u = -3u-3v+32$

$\boxed{v=8}$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = 1/4$$

and $8x+4y = 8\left(\frac{1}{4}(u+v)\right) + 4\left(\frac{1}{4}(v-3u)\right) = 2(u+v)+(v-3u)$

$= 3v-u$

Thus
$$\iint_R (8x+4y)\,dA = \int_0^8\int_{-4}^4 (3v-u)\left(\tfrac{1}{4}\right)du\,dv$$
$$= \frac{1}{4}\int_0^8 \left(3uv - \tfrac{1}{2}u^2 \Big|_{-4}^{4}\right)dv$$
$$= \frac{1}{4}\int_0^8 \left(3v(4+4) - \tfrac{1}{2}(16-16)\right)dv$$
$$= \frac{1}{4}\int_0^8 24v\,dv = 3v^2\Big|_0^8 = \boxed{192}$$

A

3. Find the position vector of a moving object that has acceleration $\vec{a}(t) = 2t\,\hat{\imath} + e^t\,\hat{\jmath} + 4\,\hat{k}$, initial velocity $\hat{\imath} + 2\,\hat{\jmath}$ and initial position $3\hat{\imath} - \hat{\jmath} + 2\,\hat{k}$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt + \vec{C}$$
$$= \int (2t\hat{\imath} + e^t\hat{\jmath} + 4\hat{k})\,dt + \vec{C}$$
$$= t^2\hat{\imath} + e^t\hat{\jmath} + 4t\hat{k} + \vec{C}$$

then $\vec{v}(0) = \hat{\jmath} + \vec{C} = \hat{\imath} + 2\hat{\jmath} \Rightarrow \vec{C} = \hat{\imath} + \hat{\jmath}$

thus $\vec{v}(t) = (t^2+1)\hat{\imath} + (e^t+1)\hat{\jmath} + 4t\hat{k}$

so $\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{Q}$
$$= \int ((t^2+1)\hat{\imath} + (e^t+1)\hat{\jmath} + 4t\hat{k})\,dt + \vec{Q}$$
$$= (\tfrac{1}{3}t^3 + t)\hat{\imath} + (e^t + t)\hat{\jmath} + 2t^2\hat{k} + \vec{Q}$$

then $\vec{r}(0) = \hat{\jmath} + \vec{Q} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k} \Rightarrow \vec{Q} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

$$\therefore \boxed{\vec{r}(t) = (\tfrac{1}{3}t^3 + t + 3)\hat{\imath} + (e^t + t - 2)\hat{\jmath} + (2t^2+2)\hat{k}}$$

(A)

4. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = z\,\hat{\imath} + y\,\hat{\jmath} - x\,\hat{k}$ and $\vec{r}(t) = t\,\hat{\imath} + \sin(t)\,\hat{\jmath} + \cos(t)\,\hat{k}$ for $0 \le t \le \pi$.

$$\vec{F}(\vec{r}(t)) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} - t\,\hat{k}$$

$$\vec{r}\,'(t) = \hat{\imath} + \cos t\,\hat{\jmath} - \sin t\,\hat{k}$$

then $\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$$= \int_0^{\pi} (\cos t\,\hat{\imath} + \sin t\,\hat{\jmath} - t\,\hat{k}) \cdot (\hat{\imath} + \cos t\,\hat{\jmath} - \sin t\,\hat{k})\,dt$$

$$= \int_0^{\pi} (\cos t + \sin t \cos t + t \sin t)\,dt$$

$$= \sin t + \frac{1}{2}\sin^2 t - t\cos t + \sin t \Big|_0^{\pi}$$

$$= 0 + 0 - \pi(-1) + 0 - 0 - 0 + 0 - 0$$

$$= \boxed{\pi}$$

$$\simeq \boxed{3.14}$$

(A)

5. Determine whether or not

$$\vec{F}(x,y) = (2x\cos(y) + e^y\cos(x))\,\hat{\imath} + (e^y\sin(x) - x^2\sin(y))\,\hat{\jmath}$$

is a conservative vector field. If it is, find its potential function.

$P(x,y) = 2x\cos y + e^y\cos x$

$Q(x,y) = e^y\sin x - x^2\sin y$

($P(x,y)$ and $Q(x,y)$ continuous and are defined on all of $\mathbb{R}^2$ as are their derivatives)

$$\left.\begin{aligned} P_y &= -2x\sin y + e^y\cos x \\ Q_x &= e^y\cos x - 2x\sin y \end{aligned}\right\} \quad \boxed{P_y = Q_x \Rightarrow \vec{F} \text{ is conservative}}$$

$f(x,y) = \int P(x,y)\,dx + g(y)$ (or $\int Q(x,y)\,dy + h(x)$)

$= \int (2x\cos y + e^y\cos x)\,dx + g(y)$

$= x^2\cos y + e^y\sin x + g(y)$

$\frac{\partial f}{\partial y} = -x^2\sin y + e^y\sin x + g'(y) = Q(x,y) = e^y\sin x - x^2\sin y$

thus $g'(y) = 0 \Rightarrow g(y) = C$

$\therefore$ the potential function is $\boxed{f(x,y) = x^2\cos y + e^y\sin x + C}$

(B)

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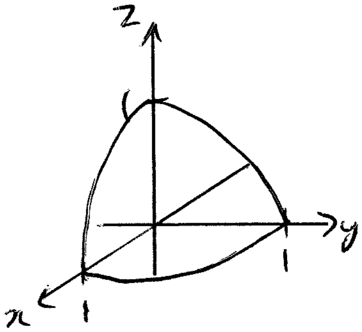
B

1. Evaluate $\displaystyle\iiint_E 9e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where E is the region inside the sphere $x^2+y^2+z^2=1$ in the first octant.



first octant $\Rightarrow$ $0 \le \phi \le \pi/2$, $0 \le \theta \le \pi/2$

$$\iiint_E 9e^{(x^2+y^2+z^2)^{3/2}}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} 9e^{\rho^3}\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\pi/2} \sin\phi\, d\phi\right)\left(\int_0^1 9\rho^2 e^{\rho^3}\, d\rho\right)$$

$$= (\pi/2)(1)\left(3e^{\rho^3}\Big|_0^1\right)$$

$$= (\pi/2)\left(3\,(e^1-1)\right)$$

$$= \boxed{\frac{3\pi}{2}(e-1)}$$

$$\simeq \boxed{8.1}$$

(B)

2. Evaluate $\displaystyle\iint_R (4x+8y)\,dA$, where R is the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$ and $(1,5)$ using the change of variable $x=\frac{1}{4}(u+v)$, $y=\frac{1}{4}(v-3u)$.

See version A for the details of the transformation

$$
\begin{aligned}
4x+8y &= 4\left(\tfrac{1}{4}(u+v)\right)+8\left(\tfrac{1}{4}(v-3u)\right)\\
&= u+v+2(v-3u)\\
&= 3v-5u
\end{aligned}
$$

$$
\begin{aligned}
\iint_R (4x+8y)\,dA &= \int_0^8\int_{-4}^4 (3v-5u)\left(\tfrac{1}{4}\right)du\,dv\\
&= \frac{1}{4}\int_0^8 \left(3uv-5u^2\Big|_{-4}^{4}\right)dv\\
&= \int_0^8 6v\,dv\\
&= 3v^2\Big|_0^8\\
&= \boxed{192}
\end{aligned}
$$

(B)

3. Find the position vector of a moving object that has acceleration $\vec{a}(t) = 4t\,\hat{\imath}+e^t\,\hat{\jmath}+6\,\hat{k}$, initial velocity $2\,\hat{\imath}-\,\hat{\jmath}$ and initial position $\hat{\imath}+3\,\hat{k}$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt + C = \int (4t\,\hat{\imath} + e^t\,\hat{\jmath} + 6\hat{k})\,dt + \vec{C}$$

$$= 2t^2\,\hat{\imath} + e^t\,\hat{\jmath} + 6t\,\hat{k} + \vec{C}$$

then $\vec{v}(0) = \hat{\jmath} + \vec{C} = 2\hat{\imath} - \hat{\jmath} \Rightarrow \vec{C} = 2\hat{\imath} - 2\hat{\jmath}$

and thus $\vec{v}(t) = (2t^2+2)\hat{\imath} + (e^t - 2)\hat{\jmath} + 6t\hat{k}$

So then $\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{Q}$

$$= \int ((2t^2+2)\hat{\imath} + (e^t-2)\hat{\jmath} + 6t\hat{k})\,dt + \vec{Q}$$

$$= \left(\tfrac{2}{3}t^3 + 2t\right)\hat{\imath} + (e^t - 2t)\hat{\jmath} + 3t^2\hat{k} + \vec{Q}$$

thus $\vec{r}(0) = \hat{\jmath} + \vec{Q} = \hat{\imath} + 3\hat{k} \Rightarrow \vec{Q} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$

$$\therefore \boxed{\vec{r}(t) = \left(\tfrac{2}{3}t^3 + 2t + 1\right)\hat{\imath} + (e^t - 2t - 1)\hat{\jmath} + (3t^2+3)\hat{k}}$$

(B)

4. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y,z) = z\,\hat{\imath} + y\,\hat{\jmath} + x\,\hat{k}$ and $\vec{r}(t) = 2t\,\hat{\imath} + \sin(t)\,\hat{\jmath} + \cos(t)\,\hat{k}$ for $0 \le t \le \pi$.

$\vec{F}(\vec{r}(t)) = \cos t\ \hat{\imath} + \sin t\ \hat{\jmath} + 2t\ \hat{k}$

$\vec{r}\,'(t) = 2\hat{\imath} + \cos t\ \hat{\jmath} - \sin t\ \hat{k}$

then $\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$

$= \int_0^{\pi} (\cos t\ \hat{\imath} + \sin t\ \hat{\jmath} + 2t\ \hat{k})\cdot(2\hat{\imath} + \cos t\ \hat{\jmath} - \sin t\ \hat{k})\,dt$

$= \int_0^{\pi} (2\cos t + \sin t\cos t - 2t\sin t)\,dt$

$= \cancel{2\sin t} + \frac{1}{2}\sin^2 t + 2t\cos t - \cancel{2\sin t}\ \Big|_0^{\pi}$

$= \frac{1}{2}\sin^2 t + 2t\cos t\ \Big|_0^{\pi}$

$= 0 + 2(\pi)(-1) - 0 - 0$

$= \boxed{-2\pi}$

$\simeq \boxed{-6.28}$

(B)

5. Determine whether or not

$$\vec{F}(x,y) = (e^x \sin(y) - y^2 \sin(x))\,\hat{\imath} + (2y\cos(x) + e^x \cos(y))\,\hat{\jmath}$$

is a conservative vector field. If it is, find its potential function.

$P(x,y) = e^x \sin y - y^2 \sin x$

$Q(x,y) = 2y\cos x + e^x \cos y$

($P(x,y)$, $Q(x,y)$ and their derivatives are defined and continuous on all of $\mathbb{R}^2$)

$P_y = e^x \cos y - 2y \sin x$

$Q_x = -2y \sin x + e^x \cos y$

$\boxed{P_y = Q_x \Rightarrow \vec{F} \text{ is conservative}}$

$f(x,y) = \int Q(x,y)\,dy + g(x)$ (or $\int P(x,y)\,dx + h(y)$)

$= \int (2y\cos x + e^x \cos y)\,dy + g(x)$

$= y^2 \cos x + e^x \sin y + g(x)$

then $\frac{\partial f}{\partial x} = -y^2 \sin x + e^x \sin y + g'(x) = P(x,y) = e^x \sin y - y^2 \sin x$

thus $g'(x) = 0 \Rightarrow g(x) = C$

$\therefore$ potential function is $\boxed{f(x,y) = y^2 \cos x + e^x \sin y + C}$