(A)

MAT 2322 C
Calculus III
Midterm # 1
February 7, 2013

Length: 80 minutes

Professor: Steve Desjardins

NAME $\boxed{\text{Solutions}}$

I.D.#

Instructions:

- Write your name and student number on this page.
- Verify that your copy of the mid-term has exactly 6 pages (including this one).
- There are 5 questions worth 4 marks each for a total of 20 points.
- You must answer all questions.
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- **You may use a simple scientific calculator. Graphing or programmable calculators are not permitted.**

A

1. Find and classify the critical points of the following function.

$$f(x, y) = x^2 - xy^2 + 2y^2$$

$f_x = 2x - y^2$ $\quad$ $f_x = 0$ if $y^2 = 2x$

$f_y = -2xy + 4y = 2y(2-x)$

$f_y = 0$ if $x = 2$ or $y = 0$

if $x = 2$, $y^2 = 4 \Rightarrow y = \pm 2 \Rightarrow (2,2)$ and $(2,-2)$

if $y = 0$, $x = 0 \Rightarrow (0,0)$

$f_{xx} = 2$, $f_{xy} = -2y$, $f_{yy} = -2x + 4$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 2(-2x+4) - (-2y)^2$

$= 8 - 4x - 4y^2$

$D(0,0) = 8$ $\quad f_{xx}(0,0) = 2 \Rightarrow$ $\boxed{(0,0) \text{ local min}}$

$D(2,2) = -16 \Rightarrow$ $\boxed{(2,2) \text{ saddle point}}$

$D(2,-2) = -16 \Rightarrow$ $\boxed{(2,-2) \text{ saddle point}}$

Ⓐ

2. Use the method of Lagrange multipliers to determine the absolute minimum and maximum values of the function $f(x, y) = x^2 - 4x + y^2$ on the circle $x^2 + y^2 = 1$.

$f(x,y) = x^2 - 4x + y^2 \qquad g(x,y) = x^2 + y^2 = 1$

$\nabla f = \lambda \nabla g \Rightarrow f_x = \lambda g_x \Rightarrow 2x - 4 = 2\lambda x$

or $x - 2 = \lambda x \Rightarrow \lambda \neq 1$

$f_y = \lambda g_y \Rightarrow 2y = 2\lambda y$

or $y = \lambda y \Rightarrow y = 0$

if $y = 0$, $x^2 = 1 \Rightarrow x = \pm 1$

so 2 points $(1,0)$ and $(-1,0)$

$\boxed{f(1,0) = -3}$

$\boxed{f(-1,0) = 5}$

$\therefore$ $\boxed{\text{min } -3}$ and $\boxed{\text{max } 5}$

A

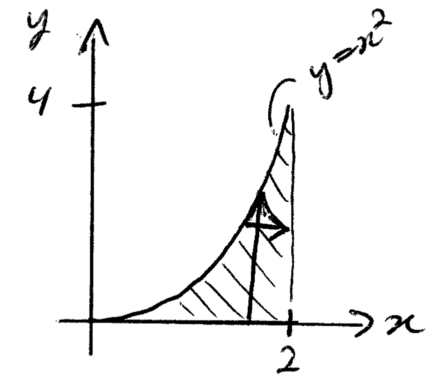
3. Calculate the following integral.

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\, dy$$

have to switch order of integration

x goes $\sqrt{y}$ to 2, then y 0 to 4

so region is



then y goes 0 to x^2
and x 0 to 2

thus $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\, dy$

$$= \int_0^2 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx$$

$$= \int_0^2 \sqrt{x^3+1} \left(y \Big|_0^{x^2} \right) dx$$

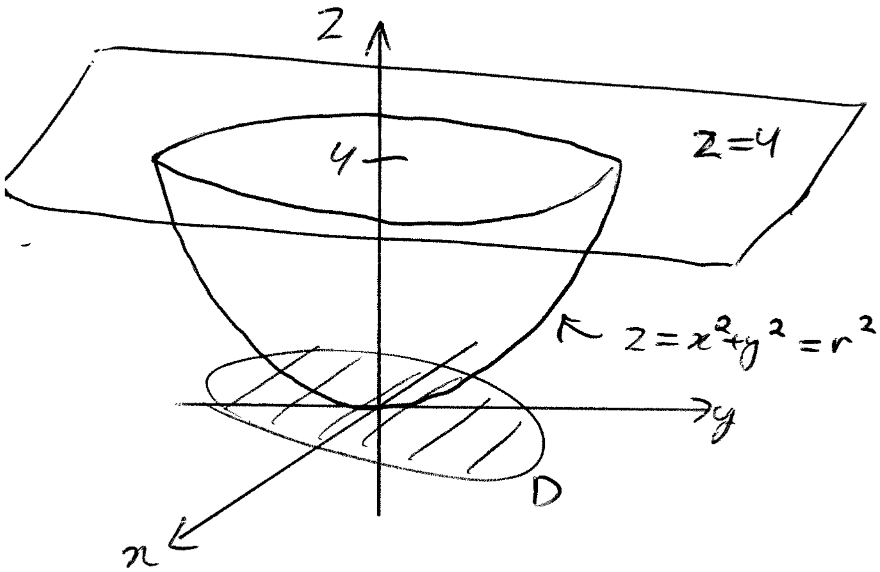
$$= \int_0^2 x^2 \sqrt{x^3+1}\, dx$$

$$= \frac{2}{9} (x^3+1)^{3/2} \Big|_0^2$$

$$= \frac{2}{9} \left((9)^{3/2} - 1 \right) = \frac{2}{9}(27-1) = \boxed{52/9} \; \boxed{\simeq 5.78}$$

A

4. Compute the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.



shadow on xy plane is disk $x^2+y^2 \leq 4$

use polar coords

So Volume is $V = \int_0^{2\pi}\int_0^2 (4-r^2)\, r\, dr\, d\theta$

$= 2\pi \int_0^2 (4r - r^3)\, dr$

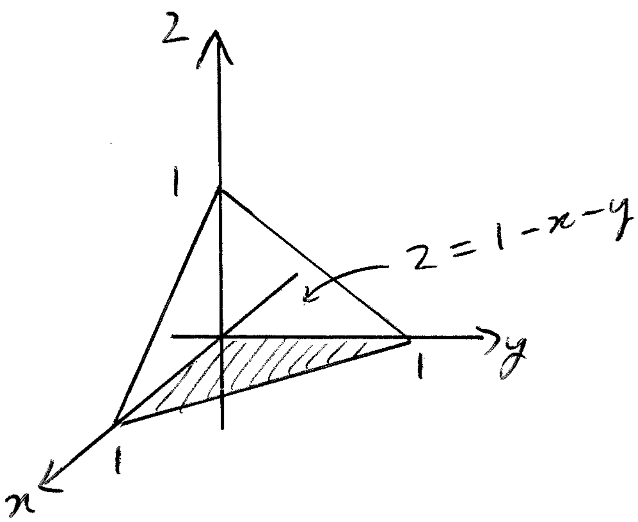
$= 2\pi \left(2r^2 - \frac{1}{4}r^4 \Big|_0^2\right)$

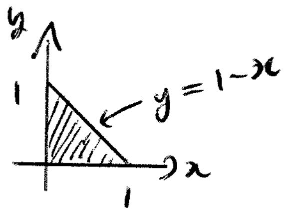
$= 2\pi\,(8-4)$

$= \boxed{8\pi} \simeq \boxed{25.13}$

Ⓐ

5. Calculate the total mass of the tetrahedron which is in the first octant and under the plane $x + y + z = 1$ where the density function is $\rho(x, y, z) = x$.





the mass is $M = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz\, dy\, dx$

$$= \int_0^1 \int_0^{1-x} x\left(z \Big|_0^{1-x-y}\right) dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (x - x^2 - xy)\, dy\, dx$$

$$= \int_0^1 \left((x - x^2) - \tfrac{1}{2} x y^2 \Big|_0^{1-x}\right) dx$$

$$= \int_0^1 \left((x - x^2)(1-x) - \tfrac{1}{2} x (1-x)^2\right) dx$$

$$= \int_0^1 \frac{1}{2} x (1-x)^2 \, dx$$

$$= \frac{1}{2} \int_0^1 (x - 2x^2 + x^3)\, dx$$

$$= \frac{1}{2} \left(\frac{1}{2} x^2 - \frac{2}{3} x^3 + \frac{1}{4} x^4 \Big|_0^1\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{2}\left(\frac{6 - 8 + 3}{12}\right) = \boxed{\frac{1}{24}} \approx \boxed{0.042}$$

(B)

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B

1. Find and classify the critical points of the following function.

$$f(x,y) = 2x^2 - x^2y + y^2$$

$f_x = 4x - 2xy = 2x(2-y)$

$f_x = 0$ if $x = 0$ or $y = 2$

$f_y = -x^2 + 2y$ $\quad f_y = 0$ if $2y = x^2$

if $x=0$, $y=0 \Rightarrow (0,0)$

if $y = 2$, $x^2 = 4 \Rightarrow x = \pm 2 \Rightarrow (2,2)$ and $(-2,2)$

$f_{xx} = 4 - 2y$, $f_{xy} = -2x$, $f_{yy} = 2$

$D = (4-2y)(2) - (-2x)^2 = 8 - 4y - 4x^2$

$D(0,0) = 8 \quad f_{xx}(0,0) = 4 \Rightarrow \boxed{(0,0) \text{ local min}}$

$D(2,2) = -16 \Rightarrow \boxed{(2,2) \text{ saddle point}}$

$D(-2,2) = -16 \Rightarrow \boxed{(-2,2) \text{ saddle point}}$

B

2. Use the method of Lagrange multipliers to determine the absolute minimum and maximum values of the function $f(x, y) = x^2 - 4y + y^2$ on the circle $x^2 + y^2 = 4$.

$f(x,y) = x^2 - 4y + y^2$, $g(x,y) = x^2 + y^2 = 4$

$\nabla f = \lambda \nabla g \Rightarrow f_x = \lambda g_x \Rightarrow 2x = 2\lambda x$

or $x = \lambda x \Rightarrow x = 0$

$f_y = \lambda g_y \Rightarrow -4 + 2y = 2\lambda y$

or $-2 + y = \lambda y \Rightarrow \lambda \neq 1$

if $x = 0$, $y^2 = 4 \Rightarrow y = \pm 2$

have 2 points $(0,2)$ and $(0,-2)$

$\boxed{f(0,2) = -4}$

$\boxed{f(0,-2) = 12}$

$\therefore$ $\boxed{\text{min } -4}$ and $\boxed{\text{max } 12}$

B

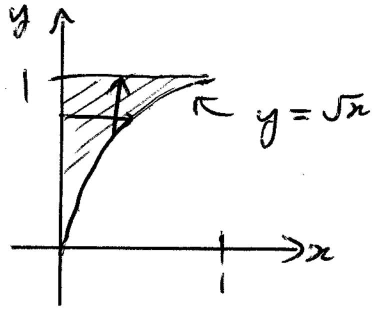
3. Calculate the following integral.

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+3}\, dy\, dx$$

have to switch order of integration

y goes $\sqrt{x}$ to 1, then x 0 to 1, so

the region is



then x goes 0 to y^2

and y 0 to 1

thus $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+3}\, dy\, dx$

$$= \int_0^1 \int_0^{y^2} \sqrt{y^3+3}\, dx\, dy$$

$$= \int_0^1 \sqrt{y^3+3}\left(x\Big|_0^{y^2}\right) dy$$

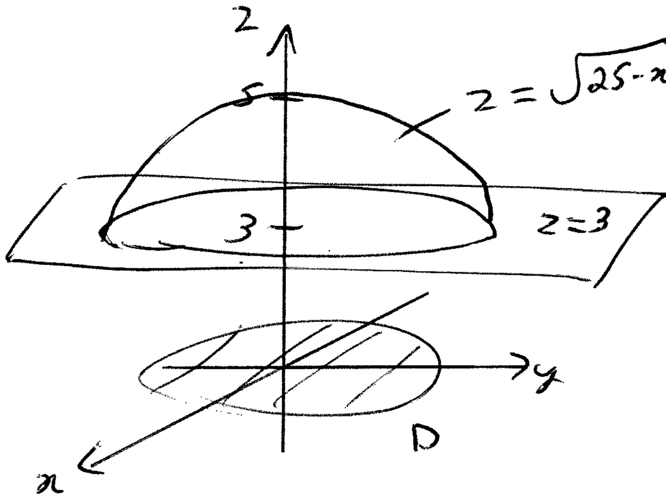
$$= \int_0^1 y^2\sqrt{y^3+3}\, dy$$

$$= \frac{2}{9}\left(y^3+3\right)^{3/2}\Big|_0^1$$

$$= \frac{2}{9}\left(4^{3/2} - 3^{3/2}\right) = \boxed{\frac{2}{9}\left(8 - 3\sqrt{3}\right)} \simeq \boxed{0.62}$$

B

4. Compute the volume of the solid bounded from above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$ and from below by the plane $z = 3$.



$= \sqrt{25-r^2}$

shadow on xy plane is disk $\sqrt{25-(x^2+y^2)} = 3$ or $x^2+y^2 \le 16$

use polar coords

the Volume is

$$V = \int_0^{2\pi}\int_0^4 \left(\sqrt{25-r^2} - 3\right) r\,dr\,d\theta$$

$$= 2\pi \int_0^4 \left(r\sqrt{25-r^2} - 3r\right) dr$$

$$= 2\pi \left(-\frac{1}{3}(25-r^2)^{3/2} - \frac{3}{2}r^2 \Big|_0^4 \right)$$

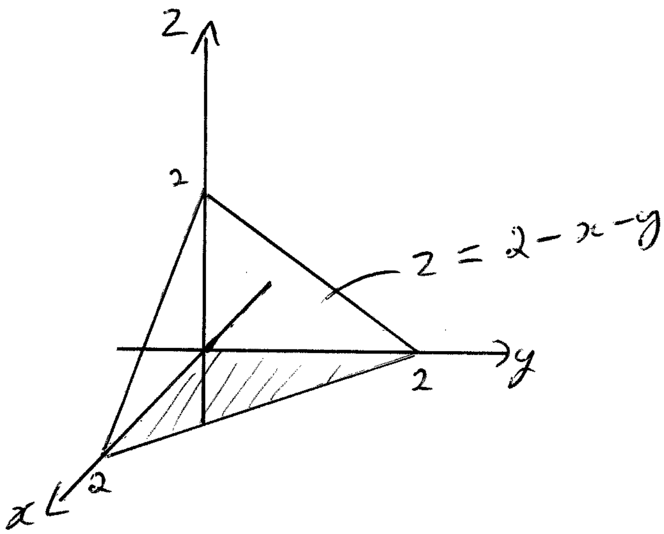
$$= 2\pi \left(-\frac{1}{3}\left((9)^{3/2} - (25)^{3/2}\right) - \frac{3}{2}(16)\right)$$

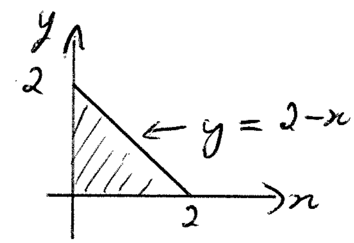
$$= 2\pi \left(-\frac{1}{3}(27 - 125) - 24\right)$$

$$= 2\pi \left(\frac{98 - 72}{3} \right)$$

$$= \boxed{52\pi/3} \simeq \boxed{54.45}$$

5. Calculate the total mass of the tetrahedron which is in the first octant and under the plane $x+y+z=2$ where the density function is $\rho(x,y,z)=z$.





the mass is $M = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} z \, dz \, dy \, dx$

$$= \int_0^2 \int_0^{2-x} \left(\frac{1}{2} z^2 \Big|_0^{2-x-y} \right) dy \, dx$$

$$= \frac{1}{2} \int_0^2 \int_0^{2-x} (2-x-y)^2 \, dy \, dx$$

$$= \frac{1}{2} \int_0^2 \left(-\frac{1}{3} (2-x-y)^3 \Big|_0^{2-x} \right) dx$$

$$= -\frac{1}{6} \int_0^2 \left((2-x-(2-x))^3 - (2-x)^3 \right) dx$$

$$= \frac{1}{6} \int_0^2 (2-x)^3 \, dx$$

$$= \frac{-1}{24} (2-x)^4 \Big|_0^2 = \frac{-1}{24}(0-16) = \boxed{\frac{2}{3}} \simeq \boxed{0.67}$$