MAT 1332, Winter 2015, Assignment 4

Due Wednesday March 11 in the math department dropboxes by 4:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Instructor (circle one): Robert Smith? Termeh Kousha Jeff Musgrave

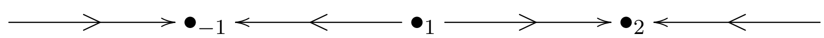
DGD (circle one): 1 2 3 4

Student Name ________________________ Student Number ________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ________________________

Question 1. Find a differential equation whose phase-line diagram is the following, and graph the solution curve for $y(0) = 3$:



[3] We'd like to create a differential equation $x' = f(x)$. The equilibrium points are $x = -1$, $x = 1$ and $x = 2$, so we need those to be roots of f. We thus have

$$x' = f(x) = a(x+1)(x-1)(x-2) = a(x^3 - 2x^2 - x + 2)$$

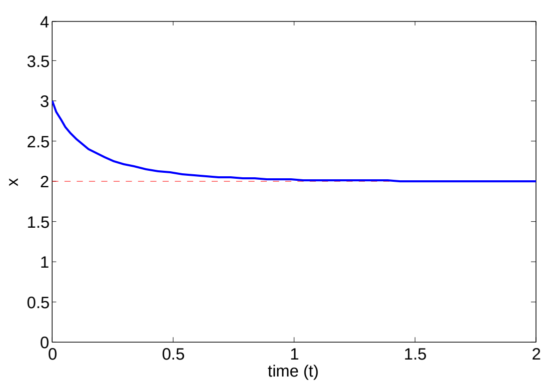
where a is to be determined (because we don't know the sign). Differentiating, we have

$$f'(x) = a(3x^2 - 4x - 1)$$
$$f'(-1) = 6a \qquad f'(1) = -2a \qquad f'(2) = 3a$$

We need $x = -1$ and $x = 2$ to be stable, which requires $f'(-1)$ and $f'(2)$ to be negative. Conversely, we need $f'(1) > 0$. These will occur if a is negative. It follows that a differential equation that has the above phase-line diagram is

$$x' = -(x+1)(x-1)(x-2)$$

The solution curve is given by the following:



QUESTION 2. Write each of the following complex numbers in the form $a+ib$, and draw them in the plane:

A: $(-1+2i)(-3+2i)$, B: $\dfrac{-1+i}{1+2i}$, C: $(2+i)^2$, D: $(1+i)^{-1}$, E: $3e^{3\pi i/2}$, F: $2e^{i\pi/3}\times 3e^{-i\pi/2}$.

(You may put them all on the same graph, but clearly indicate which is which.)

[4] We have

A: $$(-1+2i)(-3+2i) = 3-2i-6i+4i^2 = -1-8i$$

B: $$\frac{-1+i}{1+2i}\cdot\frac{1-2i}{1-2i} = \frac{-1+2i+i-2i^2}{1+4} = \frac{1+3i}{5} = \frac{1}{5}+\frac{3}{5}i$$

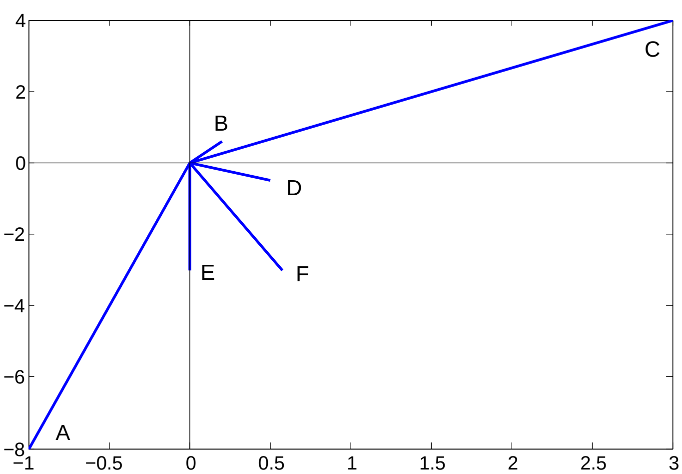
C: $$(2+i)^2 = 4+4i-1 = 3+4i$$

D: $$\frac{1}{1+i}\cdot\frac{1-i}{1-i} = \frac{1-i}{1+1} = \frac{1}{2}-\frac{1}{2}i$$

E: $$3e^{3\pi i/2} = 3\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right) = -3i$$

F: $$2e^{i\pi/3}\times 3e^{-i\pi/2} = 6e^{-i\pi/6} = 6\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right) = 3\sqrt{3}-3i = 0.577-3i$$

[B: 1 mark for the form and one for the graph.]
[E: 1 mark for the form and one for the graph.]



QUESTION 3. For the system of linear equations

$$\begin{array}{ccccccc} x & + & 3y & + & 9z & = & 3 \\ 2x & + & 7y & + & 23z & = & 2 \\ x & + & ay & + & a^2 z & = & a \end{array}$$

determine the values of a for which the system has

(i) no solution,

(ii) infinitely many solutions,

(iii) a unique solution.

[5] The augmented matrix of the system is

$$A = \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 2 & 7 & 23 & 2 \\ 1 & a & a^2 & a \end{array}\right]$$

We perform the following operations, where R_i is row i: $R_2 \rightsquigarrow R_2 - 2R_1$, $R_3 \rightsquigarrow R_3 - R_1$, $R_3 \rightsquigarrow R_3 - (a-3)R_2$, and obtain

$$A \sim \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & a-3 & a^2-9 & a-3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & a^2-9-5a+15 & 5(a-3) \end{array}\right].$$

Since $a^2 - 9 - 5a + 15 = (a-3)(a-2)$ we get :

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & (a-3)(a-2) & 5(a-3) \end{array}\right]$$

- If $a = 2$, then the last row of the matrix is is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -5 \end{array}\right]$. Hence the system is inconsistent.

- If $a = 3$ then
$$M = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -6 & 10 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
Hence the system has infinitely many solutions.

- If $a \notin \{3, 2\}$, then $(a-3)(a-2) \neq 0$ and so the system is uniquely solvable

The answer is therefore :

(i) The system in inconsistent if $a = 2$.

(ii) The system has infinitely many solutions if $a = 3$.

(iii) The system is uniquely solvable if $a \notin \{2, 3\}$.

QUESTION 4. Given the following matrices and vectors

$$A = \begin{bmatrix} 1 & -7 & -9 \\ 6 & 0 & 8 \\ 4 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & -7 \\ 6 & 4 \\ 0 & 1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 7/4 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1/2 \\ 3/4 \end{bmatrix}.$$

Compute the following if possible. If not possible, explain in one sentence why.

a) $3\mathbf{v} + 2B^T\mathbf{w}$ is not defined since $B^T\mathbf{w}$ is 2×1 and $\mathbf{v}$ is 3×1.

b) $\mathbf{w}\mathbf{v}^T = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 7/8 & 0 \\ 3/4 & 21/6 & 0 \end{bmatrix}.$

[1] **c)** $\mathbf{v}^T\mathbf{w} = 1 * 0 + 7/4 * 1/2 + 0 * 3/4 = 7/8.$

[1] **d)** $A^TB + 2\mathbf{v}^T\mathbf{w}$ is not defined since A^TB is a 3×2 matrix and $\mathbf{v}^T\mathbf{w}$ is 1×1.

e) $AB = \begin{bmatrix} -34 & 44 \\ 18 & -34 \\ 36 & -12 \end{bmatrix}.$

f) $B\mathbf{u}$ is not defined since B has 2 columns and $\mathbf{u}$ has 3 rows.

[1] **g)** BA is not defined since B has 2 columns and A has 3 rows.

[1] **h)** $A^2 = \begin{bmatrix} -77 & -43 & -65 \\ 38 & -10 & -54 \\ 28 & -28 & -4 \end{bmatrix}.$

QUESTION 5. Determine the matrix A such that:

$$\left(3A^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} -4 & 3 \\ 2 & 4 \\ -2 & 6 \end{bmatrix} + 3\begin{bmatrix} 7 & -5 & 4 \\ 9 & 12 & 3 \end{bmatrix}^T.$$

We calculate both sides of the equation

$$\left(3A^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}\right)^T = 3(A^T)^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}^T = 3A - \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 3 \\ 2 & 6 \\ -2 & 6 \end{bmatrix} + 3\begin{bmatrix} 7 & -5 & 4 \\ 9 & 12 & 3 \end{bmatrix}^T = \begin{bmatrix} -4 & 3 \\ 2 & 6 \\ -2 & 6 \end{bmatrix} + 3\begin{bmatrix} 7 & 9 \\ -5 & 12 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 2 & 6 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} 21 & 27 \\ -15 & 36 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 17 & 30 \\ -13 & 42 \\ 10 & 15 \end{bmatrix}$$

Hence

$$3A - \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 30 \\ -13 & 42 \\ 10 & 15 \end{bmatrix}$$

Therefore

$$3A = \begin{bmatrix} 17 & 30 \\ -13 & 42 \\ 10 & 6+9 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 26 \\ -10 & 47 \\ 8 & 16 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} 18 & 26 \\ -10 & 47 \\ 8 & 16 \end{bmatrix} = \begin{bmatrix} 6 & \frac{26}{3} \\ \frac{-10}{3} & \frac{47}{3} \\ \frac{8}{3} & \frac{16}{3} \end{bmatrix}.$$