MAT 1332, Winter 2015, Assignment 2

Due Wednesday February 4 in the math department dropboxes by 4:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Instructor (circle one): Robert Smith? Termeh Kousha Jeff Musgrave

DGD (circle one): 1 2 3 4

Student Name ______________________ Student Number ______________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________

QUESTION 1. For each of the following integrals do the following:
(i) Explain in one sentence why the integral is an improper integral.
(ii) Decide whether the integral is convergent or divergent. Justify your answer.

(a) $\displaystyle\int_1^3 \frac{1}{(t-3)^{4/3}}dt$ (b) $\displaystyle\int_{-1}^1 \frac{x}{\sqrt{1-x^2}}dx\,.$

[1] (i) (a)
$\displaystyle\int_1^3 \frac{1}{(t-3)^{4/3}}dt$ is an improper integral since $f(t) = \dfrac{1}{(t-3)^{4/3}}$ is continuous on $[1,3)$ but not continuous on $[1,3]$.

(i) (b)
$\displaystyle\int_{-1}^1 \frac{x}{\sqrt{1-x^2}}dx$ is an improper integral since $y(x) = \dfrac{x}{\sqrt{1-x^2}}$ is continuous on $(-1,1)$ but discontinuous on $[-1,1]$.

[2] (ii) (a)

$$\begin{aligned}\int_1^x \frac{1}{(t-3)^{4/3}}dt &= \int_1^x (t-3)^{-4/3}dt \\ &= \left[-3(t-3)^{-1/3}\right]_1^x \\ &= -3(x-3)^{-1/3} + 3(-2)^{-1/3} \\ &= -3(x-3)^{-1/3} - \frac{3}{\sqrt[3]{2}}\end{aligned}$$

for all $x \geq 1$. Taking the limit, we have

$$\lim_{x\to 3^-}\left(-3(x-3)^{-1/3} - \frac{3}{\sqrt[3]{2}}\right) = -3\lim_{x\to 3^-}(x-3)^{-1/3} - \frac{3}{\sqrt[3]{2}} = \infty$$

since $\lim_{x\to 3^-} \sqrt[3]{x-3} = 0^-$ (i.e., approaches zero from below). It follows that

$$\lim_{x\to 3^-} \int_1^x \frac{1}{(t-3)^{4/3}} dt$$

does not exist and hence the improper integral is <u>divergent</u>.

(ii) (b)

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx = \underbrace{\int_{-1}^{0} \frac{x}{\sqrt{1-x^2}} dx}_{(1)} + \underbrace{\int_{0}^{1} \frac{x}{\sqrt{1-x^2}} dx}_{(2)}$$

First we have

(1) $$\int_t^0 \frac{x}{\sqrt{1-x^2}} dx = \int_{1-t^2}^{1} \frac{-1}{2\sqrt{y}} dy \qquad y = 1-x^2, \quad \frac{dy}{dx} = -2x$$

$$= -\sqrt{y}\Big|_{1-t^2}^{1} = -\left[\sqrt{1} - \sqrt{1-t^2}\right] = -1 + \sqrt{1-t^2}$$

Hence

$$\lim_{t\to -1} \int_t^0 \frac{x}{\sqrt{1-x^2}} dx = \lim_{t\to -1} \left[-1 + \sqrt{1-t^2}\right] = -1$$

Next we have

(2) $$\int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \int_1^{1-t^2} \frac{-1}{2\sqrt{y}} dy \qquad y = 1-x^2, \quad \frac{dy}{dx} = -2x$$

$$= -\sqrt{y}\Big|_{1}^{1-t^2} = -\left[\sqrt{1-t^2} - 1\right] = 1 - \sqrt{1-t^2}$$

Hence

$$\lim_{t\to 1} \int_0^t \frac{x}{\sqrt{1-x^2}} dx = \lim_{t\to 1} \left[1 - \sqrt{1-t^2}\right] = 1$$

Since both (1) and (2) are convergent, it follows that $\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx$ is <u>convergent</u> and

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx = -1 + 1 = 0$$

QUESTION 2. Evaluate the following indefinite integrals:

$$\text{(a)} \int \frac{x^3 + x^2 + 5}{x^2 + 4} dx \qquad \text{(b)} \int \frac{x^4 - 8x^2 - 10}{x^2 - 3x - 10} dx$$

(a)

Since the degree of the numerator is larger than the degree of the denominator, we need to do long division

$$\begin{array}{r|l} & x + 1 \\ \hline x^2 + 4 & x^3 + x^2 + 5 \\ & x^3 + 4x \\ \hline & x^2 - 4x + 5 \\ & x^2 + 4 \\ \hline & -4x + 1 \end{array}$$

Hence $x^3 + x^2 + 5 = (x + 1)(x^2 + 4) - 4x + 1$ and so

$$\begin{aligned} \frac{x^3 + x^2 + 5}{x^2 + 4} &= \frac{(x + 1)(x^2 + 4) - 4x + 1}{x^2 + 4} \\ &= x + 1 + \frac{1 - 4x}{x^2 + 4} \end{aligned}$$

Note that $x^2 + 4$ cannot be factored and hence

$$x + 1 + \frac{1}{x^2 + 4} - \frac{4x}{x^2 + 4}$$

is the partial fraction decomposition. We thus have

$$\begin{aligned} \int \frac{x^3 + x^2 + 5}{x^2 + 4} dx &= \int (x + 1) dx + \int \frac{1}{x^2 + 4} dx - \int \frac{4x}{x^2 + 4} dx \\ &= \frac{1}{2} x^2 + x + \frac{1}{2} \arctan\left(\frac{x}{2}\right) - 2\ln(x^2 + 4) + C \end{aligned}$$

(Note that we don't need the absolute value signs in the logarithm because $x^2 + 4$ is automatically positive.)

[6] (b)

Once again, we need to do long division:

$$\begin{array}{r|l} & x^2 + 3x + 11 \\ \hline x^2 - 3x - 10 & x^4 - 8x^2 - 10 \\ & x^4 - 3x^3 - 10x^2 \\ \hline & 3x^3 + 2x^2 - 10 \\ & 3x^3 - 9x^2 - 30x \\ \hline & 11x^2 + 30x - 10 \\ & 11x^2 - 33x - 110 \\ \hline & 63x + 100 \end{array}$$

Hence $x^4 - 8x^2 - 10 = (x^2 - 3x - 10)(x^2 + 3x + 11) + 63x + 100$, so we have

$$\frac{x^4 - 8x^2 - 10}{x^2 - 3x - 10} = x^2 + 3x + 11 + \frac{63x + 100}{x^2 - 3x - 10}$$

Note that the denominator factors as $x^2 - 3x - 10 = (x + 2)(x - 5)$. Hence, using partial fractions, we have

$$\begin{aligned} \frac{63x + 100}{x^2 - 3x - 10} &= \frac{A}{x + 2} + \frac{B}{x - 5} \\ 63x + 100 &= A(x - 5) + B(x + 2) \end{aligned}$$

$$x = 5 \qquad 415 = B(7) \qquad B = \frac{415}{7}$$

$$x = -2 \qquad -26 = A(-7) \qquad A = \frac{26}{7}$$

Thus

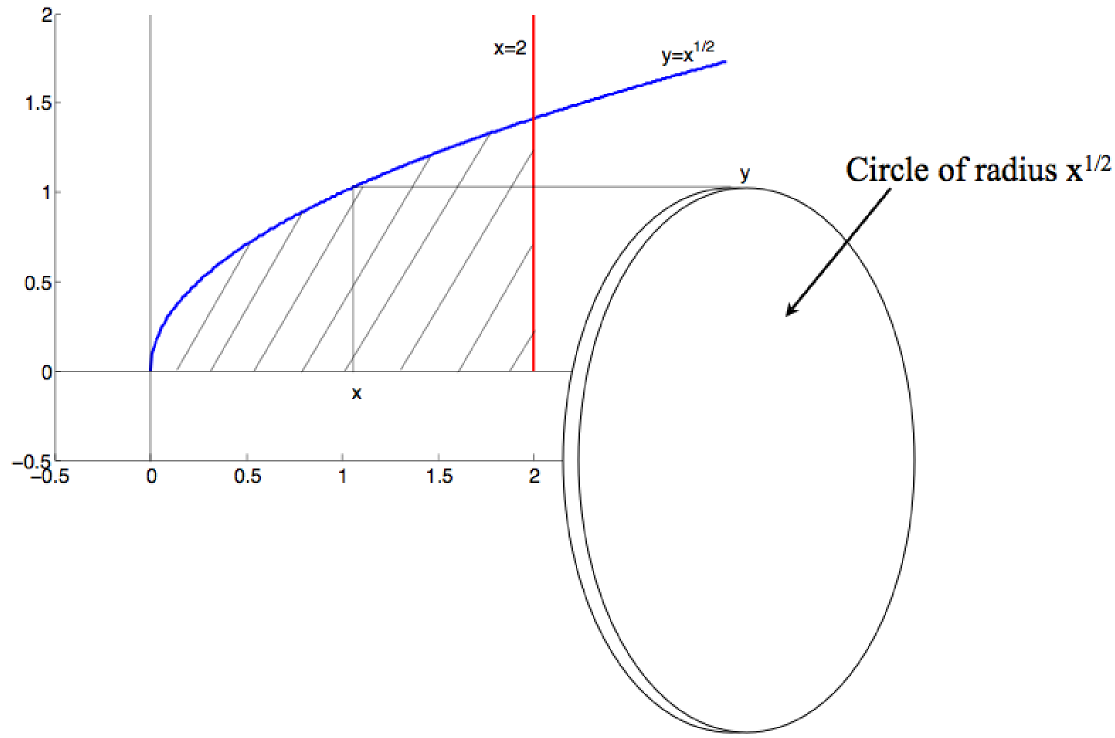
$$\begin{aligned} \int \frac{x^4 - 8x^2 - 10}{x^2 - 3x - 10} dx &= \int (x^2 + 3x + 11) dx + \int \frac{26}{7(x + 2)} + \int \frac{415}{7(x - 5)} dx \\ &= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 11x + \frac{26}{7} \ln |x + 2| + \frac{415}{7} \ln |x - 5| + C \end{aligned}$$

[Note: you lose half a mark each time you forget the absolute value signs in a logarithm and another half a mark if you forget the $+C$.]

Question 3. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$, $x = 2$ and $y = 0$ about the x-axis. Include a sketch of the region and a typical cross-section with a plane orthogonal to the x-axis. Leave your answer in exact form (not a decimal approximation).

The volume is

$$V = \int_0^2 \pi f(x)^2 dx = \int_0^2 \pi \left(\sqrt{x}\right)^2 dx = \int_0^2 \pi x dx = \frac{\pi}{2} x^2 \Big|_0^2 = \frac{\pi}{2}(4 - 0) = 2\pi \text{ units}^3$$



QUESTION 4.

a) Find the area of the region bounded by the curves $y = 2 - x$, $y = x^2$ and the line $y = 0$. (You may assume $x \geq 0$.) Include a sketch of the area. Leave your answers in exact form (not a decimal approximation).

b) Find the volume of the solid obtained by revolving the area in part (a) around the x-axis. Leave your answer in exact form.

[3] (a)

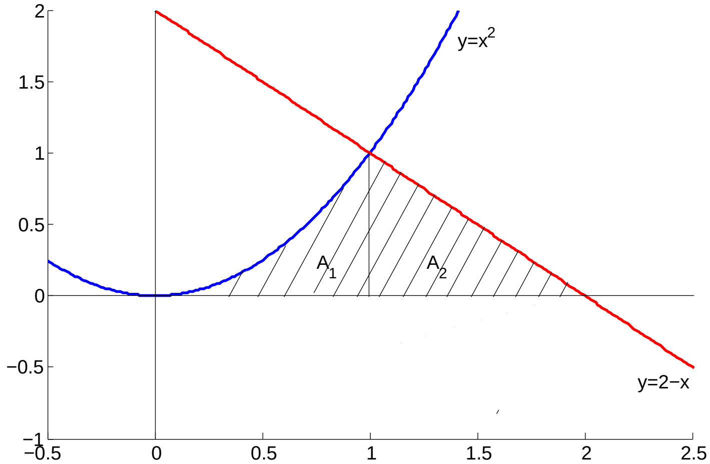
The curves $y = x^2$ and $y = 2 - x$ intersect at $(x, y) = (1, 1)$. Thus the area is

$$A = A_1 + A_2$$

where $A_1 = \int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$

$$A_2 = \int_1^2 (2-x)dx = \left[2x - \frac{1}{2}x^2\right]_1^2 = \left[4 - \frac{1}{2}\cdot 4 - 2 + \frac{1}{2}\right] = \frac{8-4-4+1}{2} = \frac{1}{2}$$

$$\therefore A = \frac{1}{3} + \frac{1}{2} = \frac{2+3}{6} = \frac{5}{6} \text{ units}^2$$



[2] (b)

The volume is

$$
\begin{aligned}
V &= \pi \int f(x)^2 dx \\
&= \pi \int_0^1 \left(x^2\right)^2 dx + \pi \int_1^2 (2-x)^2 dx \\
&= \pi \int_0^1 x^4 dx + \pi \int_1^2 (4-4x+x^2) dx \\
&= \pi \frac{x_5}{5}\Big|_0^1 + \pi \left[4x - 2x^2 + \frac{x^3}{3}\right]_1^2 \\
&= \frac{\pi}{5} + \pi \left[\left(8-8+\frac{8}{3}\right) - \left(4-2+\frac{1}{3}\right)\right] \\
&= \frac{\pi}{5} + \frac{\pi}{3} = \frac{8\pi}{15} \text{ units}^3
\end{aligned}
$$

QUESTION 5. A bacterial culture contains 500 units at the beginning. The culture grows at a rate proportional to the size of the culture. After 3 hours, the bacterial culture has grown to 800 units.

a) Find the amount of bacteria as a function of time (in hours).

b) How many units of bacteria are there after 4 hours?

c) How long does it take until the amount of bacteria reaches 30,000 units?

(a)

The differential equation that describes this phenomenon is

$$\frac{dB}{dt} = kB$$

Separating variables, we have

$$\begin{aligned}
\frac{dB}{B} &= kdt \\
\int \frac{dB}{B} &= \int kdt \\
\ln B &= kt + c \\
B &= B_0 e^{kt} && \text{(where } B_0 = e^c\text{)}
\end{aligned}$$

Applying the initial condition, we have

$$\begin{aligned}
500 &= B_0 e^0 \\
B &= 500 e^{kt}
\end{aligned}$$

After 3 hours, we have

$$\begin{aligned}
800 &= 500 e^{3k} \\
3k &= \ln \frac{8}{5} \\
k &= \frac{1}{3} \ln \frac{8}{5} \\
B(t) &= 500 e^{\frac{1}{3} \ln \frac{8}{5} t}
\end{aligned}$$

(b)
After 4 hours, we have

$$B(4) = 500 e^{\frac{4}{3} \ln \frac{8}{5}} = 500 \left(\frac{8}{5} \right)^{4/3} = 935.69 \text{ units}$$

(c)
We have

$$\begin{aligned}
30,000 &= 500 e^{\frac{1}{3} \ln \frac{8}{5} \bar{t}} \\
60 &= e^{\frac{1}{3} \ln \frac{8}{5} \bar{t}} \\
\frac{1}{3} \ln \frac{8}{5} \bar{t} &= \ln 60 \\
\ln \frac{8}{5} \bar{t} &= 3 \ln 60 \\
\bar{t} &= \frac{3 \ln 60}{\ln \frac{8}{5}} = 26.13 \text{ hours}
\end{aligned}$$

[Note: you lose half a mark if you don't specify the units.]