MAT 1332, Winter 2015, Assignment 6

Due Monday April 13 in the math department dropboxes by 9:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Instructor (circle one): Robert Smith? Termeh Kousha Jeff Musgrave

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Student Name ______________________________ Student Number ______________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________________

QUESTION 1. Consider the function $g(x, y) = \sqrt{9 - x^2 - y^2}$.

(a) Determine the domain and range of g.
Domain: we have

$$D_f = \{(x, y) \in \mathbb{R}^2 |\ 9 - x^2 - y^2 \geq 0\} = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 9\}\,.$$

Range: $R_f = [0, 3]$

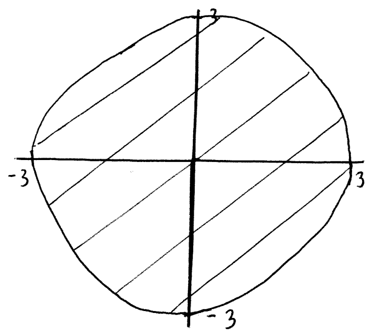
(b) Sketch the domain found in (a) in the xy-plane.
See the figure below.

(c) Sketch level curves of g for $k = 0, 1, 2$.
For all $k \in R_f$ we have

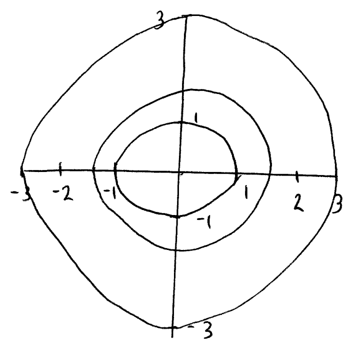
$$f(x, y) = k \Leftrightarrow \sqrt{9 - x^2 - y^2} = k \Leftrightarrow x^2 + y^2 = 9 - k^2\,.$$

So the level curves are circles with radius $\sqrt{9 - k^2}$.
When $k = 0$ we obtain $x^2 + y^2 = 9$, a circle with radius 3;
When $k = \sqrt{5}$ we obtain $x^2 + y^2 = 2$, a circle with radius $\sqrt{2}$;
When $k = \sqrt{8}$ we obtain $x^2 + y^2 = 1$, a circle with radius 1.
See the figure below.

1) b)



c)



(d) Determine the equation of the tangent plane to g at the point $(\sqrt{3}, \sqrt{2})$.
The equation of the tangent plane to f at (x_0, y_0) is given by

$$T(x, y) = g(x_0, y_0) + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0)$$

The partial derivatives of $g(x, y)$ are

$$g_x = \frac{-x}{\sqrt{9 - x^2 - y^2}}, \qquad g_y = \frac{-y}{\sqrt{9 - x^2 - y^2}}$$

and the tangent plane at $(\sqrt{3}, \sqrt{2})$ is given by

$$T(x, y) = 2 - \frac{\sqrt{3}}{2}(x - \sqrt{3}) - \frac{1}{\sqrt{2}}(y - \sqrt{2}).$$

QUESTION 2. Find the partial derivatives $\frac{\partial f}{\partial x}$ (or f_x) and $\frac{\partial f}{\partial y}$ (or f_y) of the function

$$f(x, y) = \frac{xy}{x - 2y}$$

and evaluate the derivatives at the point $(3, 1)$.
We have the following

$$f_x = \frac{-2y^2}{(x - 2y)^2}$$

$$f_y = \frac{x^2}{(x - 2y)^2}$$

Thus, $f_x(3, 1) = \dfrac{-2(1)^2}{((3) - 2(1))^2} = -2$ and $f_y(3, 1) = \dfrac{(3)^2}{((3) - 2(1))^2} = 9$.

QUESTION 3. Consider the vector valued function

$$F(x, y) = \begin{bmatrix} \sin(y) - \cos(x) \\ yx - x^2 \end{bmatrix}.$$

Find the Jacobian matrix, evaluate it at the point $(x_0, y_0) = (\frac{\pi}{2}, \frac{\pi}{2})$ and find the linear approximation at that point.
The Jacobian matrix is

$$\begin{bmatrix} \sin(x) & \cos(y) \\ y - 2x & x \end{bmatrix}$$

and evaluated at the point $(\frac{\pi}{2}, \frac{\pi}{2})$ we get the matrix

$$\begin{bmatrix} 1 & 0 \\ -\frac{\pi}{2} & \frac{\pi}{2} \end{bmatrix}$$

The linear approximation of $F(x, y)$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is

$$L(x, y) = F\left(\frac{\pi}{2}, \frac{\pi}{2}\right) + \begin{bmatrix} 1 & 0 \\ -\frac{\pi}{2} & \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} x - \frac{\pi}{2} \\ y - \frac{\pi}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} x - \frac{\pi}{2} \\ \frac{\pi}{2}(y - x) \end{bmatrix}$$

$$= \begin{bmatrix} 1 + x - \frac{\pi}{2} \\ \frac{\pi}{2}(y - x) \end{bmatrix}$$

QUESTION 4. Consider the following system of linear differential equations:

$$\frac{dx}{dt} = 11x - 3y$$
$$\frac{dy}{dt} = 26x - 8y$$

(a) Find the eigenvalues and eigenvectors associated with the system.
The coefficient matrix, A, of the above system is

$$A = \begin{pmatrix} 11 & -3 \\ 26 & -8 \end{pmatrix}$$

The eigenvalues of A satisfy $P_A(\lambda) = \det(A - \lambda I_2) = 0$. We have

$$\begin{aligned} \det(A - \lambda I_2) &= \begin{vmatrix} 11-\lambda & -3 \\ 26 & -8-\lambda \end{vmatrix} \\ &= (11-\lambda)(-8-\lambda) + 78 = \lambda^2 - 3\lambda - 10 = 0 \end{aligned}$$

Therefore, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -2$.

For each of these eigenvalues we solve $(A - \lambda I_2)\vec{x} = \vec{0}$.

- For $\lambda = 5$ we have

$$A - 5I_2 = A = \begin{bmatrix} 6 & -3 \\ 26 & -13 \end{bmatrix} \xrightarrow{R_2/13} \begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_2-(1/3)R_1} \begin{bmatrix} 6 & -3 \\ 0 & 0 \end{bmatrix}.$$

An eigenvector is then $\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

- For $\lambda = -2$ we have

$$A - (-2)I_2 = \begin{bmatrix} 13 & -3 \\ 26 & -6 \end{bmatrix} \xrightarrow{R_2-(1/2)R_1} \begin{bmatrix} 13 & -3 \\ 0 & 0 \end{bmatrix}.$$

An eigenvector is then $\vec{v}_2 = \begin{bmatrix} 1 \\ 13/3 \end{bmatrix}$.

(b) Find and classify the equilibrium, and determine its stability.
The only equilibrium of a linear system is $(0, 0)$. You should verify this on your own. Since one of the eigenvalues is positive, the equilibrium is unstable

(c) Draw the x and y-nullclines in the phase plane. On each nullcline, draw at least one direction arrow.

The x-nullcline is found by solving $\frac{dx}{dt} = 0$ and we obtain $y = \frac{11}{3}x$.

Along the x-nullcline, we have $\frac{dy}{dt} = 26x - 8(11/3)x = -\frac{10}{3}x$.

The y-nullcline is found by solving $\frac{dy}{dt} = 0$ and we obtain $y = \frac{13}{4}x$.

Along the y-nullcline, we have $\frac{dx}{dt} = 11x - 3(13/4)x = \frac{5}{4}x$

The nullclines and direction arrows are shown in the figure below.

(d) Find the particular solution satisfying $x(0) = 2, y(0) = 7$.

The general solution is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = a_1 e^{5t}\vec{v}_1 + a_2 e^{-2t}\vec{v}_2 = \begin{bmatrix} \frac{a_1}{2}e^{5t} + a_2 e^{-2t} \\ a_1 e^{5t} + \frac{13a_2}{3}e^{-2t} \end{bmatrix} \; a_1,\, a_2 \text{ are constants}.$$

The particular solution which satisfies the initial condition $x(0) = 2, y(0) = 7$ is

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{a_1}{2} + a_2 \\ a_1 + \frac{13a_2}{3} \end{bmatrix}$$

which gives

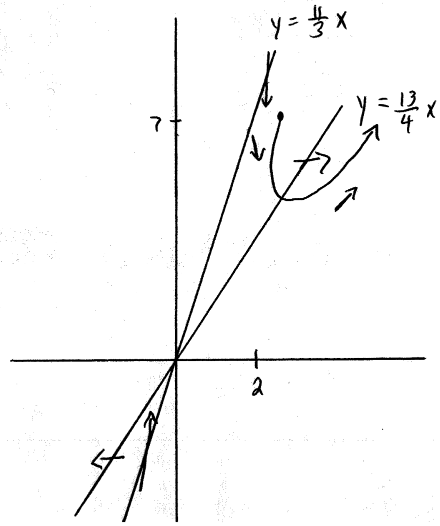
$$\begin{cases} \frac{a_1}{2} + a_2 & = & 2 \\ a_1 + \frac{13a_2}{3} & = & 7 \end{cases}.$$

We obtain $a_1 = 10/7$ and $a_2 = 9/7$ and the unique solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{5}{7}e^{5t} + \frac{9}{7}e^{-2t} \\ \frac{5}{7}e^{5t} + \frac{39}{7}e^{-2t} \end{bmatrix}$$

(e) Sketch the solution curve for the initial value $x(0) = 2, y(0) = 7$ in the phase plane.

4)



QUESTION 5. Consider a disease that propogates according to the system

$$\frac{dx}{dt} = 16 - 0.2xy - 0.4x$$
$$\frac{dy}{dt} = 0.1xy - 8y$$

where x represents susceptible individuals and y represents infected individuals.

(a) Find all biologically meaningful steady states.
Biologically meaningful here simply means that the numbers are not negative. The steady states, or equilibrium points, are the places where both $16 - 0.2xy - 0.4x = 0$ and $0.1xy - 8y = 0$. The second equation is easier (since we can factor it) so we deal with it first: $y(0.1x - 8) = 0$ when $y = 0$ or when $x = 8/0.1 = 80$. For each of these cases we plug the given value into the first equation (which must also hold).
If $y = 0$ then the first equation says that $16 - 0.4x = 0$, so $x = 16/0.4 = 40$. Therefore (40,0) is one equilibrium.
The only other case is when $x = 80$. Here, the first equation says that $16 - 0.2(80)y - 0.4(80) = 0$, so $16y = -16$ and $y = -1$. Therefore (80,-1) is another equilibrium, and there are no others. This equilibrium point is not biologically meaningful since its second coordinate is negative.

(b) Find the Jacobian matrix.
The Jacobian matrix of

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 16 - 0.2xy - 0.4x \\ 0.1xy - 8y \end{pmatrix}$$

is given by

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

We just have to conrm four partial derivatives were given correctly. So, for example,

$$\frac{\partial}{\partial x}(16 - 0.2xy - 0.4x) = -0.4 - .2y$$

(c) For the biologically meaningful steady states from (a), find the eigenvalues of the Jacobian matrix.

We have one biologically meaningful steady states: (40,0). We plug $x = 40$, $y = 0$ into the formula given in part (b) for J:

$$J(40, 0) = \begin{pmatrix} -0.4 & -8 \\ 0 & -4 \end{pmatrix}$$

This matrix is upper-triangular (since the only entry below the main diagonal is zero), so its eigenvalues are its diagonal entries: 0.4 and 4. You should compute the characteristic polynomial to verify that these are in fact the eigenvalues.

(d) Determine the stability of the biologically meaningful steady states. What is the long-term effect of the disease?

Since the eigenvalues of the Jacobian matrix at the equilibrium have negative real part (in fact, are negative real numbers), we can conclude that this equilibrium is stable. In fact, it is a stable sink.

What this means in concrete terms is that starting from any population with any infection rate, after enough time the end result will be that x is very close to 40 and y is very close to 0; in particular the disease will be wiped out in time.

(e) Find and sketch the nullclines in the region $x \geq 0$, $y \geq 0$. You must indicate the directions on the nullclines and the regions divided by the nullclines.

5) x-nullcline: $\dot{x} = 0 \Rightarrow 16 - 0.2xy - 0.4x = 0$

$$\Rightarrow y = \frac{80}{x} - 2$$

Along the x-nullcline,

$$\dot{y} = 0.1x\left(\frac{80}{x} - 2\right) - 8\left(\frac{80}{x} - 2\right)$$

$$\dot{y} = \left(\frac{80}{x} - 2\right)(0.1x - 8)$$

$$\dot{y} = \frac{1}{10}\left(\frac{80}{x} - 2\right)(x - 80)$$

y-nullcline: $\dot{y} = 0 \Rightarrow y(0.1x - 8) = 0$

$$\Rightarrow y = 0 \text{ or } x = 80$$

Along the y-nullcline $y = 0$, $\dot{x} = 16 - 0.4x$

$x = 80$, $\dot{x} = -16y - 16$